Geometry of Shimura varieties of Hodge type over finite fields

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Abstract. We present a general and comprehensive overview of recent developments in the theory of integral models of Shimura varieties of Hodge type. The paper covers the following topics: construction of integral models, their possible moduli interpretations, their uniqueness, their smoothness, their properness, and basic stratifications of their special fibres.

Keywords. Abelian and semiabelian schemes, Mumford–Tate groups, Shimura varieties, Hodge cycles, integral models, Néron models, p-divisible groups, F-crystals, and stratifications.

1. Introduction

This paper is an enlarged version of the three lectures we gave in July 2007 during the summer school *Higher dimensional geometry over finite fields*, June 25 - July 06, 2007, Mathematisches Institut, Georg-August-Universität Göttingen.

The goal of the paper is to provide to non-specialists an *efficient, accessible, and in depth* introduction to the theory of *integral models* of *Shimura varieties of Hodge type*. Accordingly, the paper will put a strong accent on defining the main objects of interest, on listing the main problems, on presenting the main techniques used in approaching the main problems, and on stating very explicitly the main results obtained so far. This is not an easy task, as only to be able to list the main problems one requires a good comprehension of the *language of schemes*, of *reductive groups*, of *abelian varieties*, of *Hodge cycles* on abelian varieties, of *cohomology theories* (including *étale and crystalline* ones), of *deformation theories*, of *p-divisible groups*, and of *F-crystals*. Whenever possible, proofs are included.

We begin with a motivation for the study of Shimura varieties of Hodge type. Let X be a connected, smooth, projective variety over $\mathbb{C}$. We recall that the *albanese variety* of X is an abelian variety $\mathrm{Alb}(X)$ over $\mathbb{C}$ equipped with a morphism $a_X : X \to \mathrm{Alb}(X)$ that has the following universal property. If $b_X : X \to B$ is another morphism from X to an abelian variety B over $\mathbb{C}$, then there exists a unique morphism $c : \mathrm{Alb}(X) \to B$ such that the following identity $b_X = c \circ a_X$ holds. This universal property determines $\mathrm{Alb}(X)$ uniquely up to isomorphisms. Not only $\mathrm{Alb}(X)$ is uniquely determined by X, but also the

image $\mathrm{Im}(a_X)$ is uniquely determined by X up to isomorphisms. Thus to X one associates an abelian variety $\mathrm{Alb}(X)$ over $\mathbb{C}$ as well as a closed subvariety $\mathrm{Im}(a_X)$ of it. If X belongs to a good class $\mathfrak{C}$ of connected, smooth, projective varieties over $\mathbb{C}$, then deformations of X would naturally give birth to deformations of the closed embedding $\mathrm{Im}(a_X) \hookrightarrow \mathrm{Alb}(X)$. Thus the study of moduli spaces of objects of the class $\mathfrak{C}$ is very much related to the study of moduli spaces of abelian schemes endowed with certain closed subschemes (which naturally give birth to some polarizations). For instance, if X is a curve of positive genus, then $\mathrm{Alb}(X) = \mathrm{Jac}(X)$ and the morphism a_X is a closed embedding; to this embedding one associates naturally a principal polarization of $\mathrm{Jac}(X)$. This implies that different moduli spaces of geometrically connected, smooth, projective curves are subspaces of different moduli spaces of principally polarized abelian schemes.

For the sake of generality and flexibility, it does not suffice to study moduli spaces of abelian schemes endowed with polarizations and with certain closed subschemes. More precisely, one is naturally led to study moduli spaces of polarized abelian schemes endowed with *families* of *Hodge cycles*. They are called *Shimura varieties of Hodge type* (see Subsection 3.4). The classical *Hodge conjecture* predicts that each Hodge cycle is an *algebraic cycle*. Thus we refer to Subsection 2.5 for a quick introduction to Hodge cycles on abelian schemes over reduced $\mathbb{Q}$–schemes. Subsections 2.1 to 2.5 review basic properties of algebraic groups, of Hodge structures, and of families of tensors.

Shimura varieties can be defined abstractly via few axioms due to Deligne (see Subsection 3). They are in natural bijection to *Shimura pairs* $(G, \mathcal{X})$. Here G is a reductive group over $\mathbb{Q}$ and X is a hermitian symmetric domain whose points form a $G(\mathbb{R})$-conjugacy class of homomorphisms $(\mathbb{C} \setminus \{0\}, \cdot) \to G_{\mathbb{R}}$ of real groups, that are subject to few axioms. Initially one gets a complex Shimura variety $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}$ defined over $\mathbb{C}$ (see Subsection 3.1). The totally discontinuous, locally compact group $G(\mathbb{A}_f)$ acts naturally on $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}$ from the right. Cumulative works of Shimura, Taniyama, Deligne, Borovoi, Milne, etc., have proved that $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}$ has a *canonical model* $\mathrm{Sh}(G, \mathcal{X})$ over a number field $E(G, \mathcal{X})$ which is intrinsically associated to the Shimura pair $(G, \mathcal{X})$ and which is called the *reflex field* of $(G, \mathcal{X})$ (see Subsection 3.2). One calls $\mathrm{Sh}(G, \mathcal{X})$ together with the natural right action of $G(\mathbb{A}_f)$ on it, as the Shimura variety defined by the Shimura pair $(G, \mathcal{X})$. For instance, if $G = \mathbf{GL}_2$ and $\mathcal{X} \tilde{\to} \mathbb{C} \setminus \mathbb{R}$ is isomorphic to two copies of the upper half-plane, then $\mathrm{Sh}(G, \mathcal{X})$ is the *elliptic modular variety* over $\mathbb{Q}$ and is the projective limit indexed by $N \in \mathbb{N}$ of the *affine modular curves* $Y(N)$.

Let H be a compact, open subgroup of $G(\mathbb{A}_f)$. The quotient scheme $\mathrm{Sh}(G, \mathcal{X})/H$ exists and is a normal, quasi-projective scheme over $E(G, \mathcal{X})$. If v is a prime of $E(G, \mathcal{X})$ of residue field $k(v)$ and if $\mathcal{N}$ is a good integral model of $\mathrm{Sh}(G, \mathcal{X})/H$ over the local ring $O_{(v)}$ of v, then one gets a Shimura variety $\mathcal{N}_{k(v)}$ over the finite field $k(v)$. The classical example of a good integral model is *Mumford moduli scheme* $\mathcal{A}_{r,1}$. Here $r \in \mathbb{N}$, the $\mathbb{Z}$-scheme $\mathcal{A}_{r,1}$ is the *course moduli scheme* of principally polarized abelian scheme of relative dimension r, and the $\mathbb{Q}$–scheme $\mathcal{A}_{r,1,\mathbb{Q}}$ is of the form $\mathrm{Sh}(G, \mathcal{X})/H$ for $(G, \mathcal{X})$ a Shimura pair that defines (see Example 3.1.2) a *Siegel modular variety*.

In this paper, we are mainly interested in Shimura varieties of Hodge type. Roughly speaking, they are those Shimura varieties for which one can naturally

choose $\mathcal{N}$ to be a finite scheme over $\mathcal{A}_{r,1,O_{(v)}}$. In this paper we study $\mathcal{N}$ and its special fibre $\mathcal{N}_{k(v)}$. See Subsections 4.1 and 4.2 for some moduli interpretations of $\mathcal{N}$. See Section 5 for different results pertaining to the uniqueness of $\mathcal{N}$. See Section 6 for basic results that pertain to the smooth locus of $\mathcal{N}$. See Section 7 for the list of cases in which $\mathcal{N}$ is known to be (or it is expected to be) a projective $O_{(v)}$-scheme. Section 8 presents four main stratifications of the (smooth locus of the) special fibre $\mathcal{N}_{k(v)}$ and their basic properties. These four stratifications are defined by (see Subsections 8.3, 8.4, 8.6, and 8.7 respectively):

(a) Newton polygons of p-divisible groups;

(b) isomorphism classes of principally quasi-polarized F-isocrystals with tensors;

(c) inner isomorphism classes of the reductions modulo integral powers of p of principally quasi-polarized F-crystals with tensors;

(d) isomorphism classes of principally quasi-polarized F-crystals with tensors.

The principally quasi-polarized F-crystals with tensors attached naturally to points of the smooth locus of $\mathcal{N}_{k(v)}$ with values in algebraically closed fields are introduced in Subsection 8.1. Generalities on stratifications of reduced schemes over fields are presented in Subsection 8.2. Subsection 8.5 shows that the smooth locus of $\mathcal{N}_{k(v)}$ is a *quasi Shimura p-variety of Hodge type* in the sense of [Va5, Def. 4.2.1]. Subsection 8.5 is used in Subsections 8.6 and 8.7 to define the last two stratifications, called the *level m* and *Traverso stratifications*.

2. A group theoretical review

In this section we review basic properties of algebraic groups, of Hodge structure, of families of tensors, and of Hodge cycles on abelian schemes over reduced $\mathbb{Q}$–schemes. We denote by $\bar{k}$ an algebraic closure of a field k.

We denote by $\mathbb{G}_a$ and $\mathbb{G}_m$ the affine, smooth groups over k with the property that for each commutative k-algebra C, the groups $\mathbb{G}_a(C)$ and $\mathbb{G}_m(C)$ are the additive group of C and the multiplicative group of units of C (respectively). As schemes, we have $\mathbb{G}_a = \mathrm{Spec}(k[x])$ and $\mathbb{G}_m = \mathrm{Spec}(k[x][\frac{1}{x}])$. Thus the dimension of either $\mathbb{G}_a$ or $\mathbb{G}_m$ is 1. For $t \in \mathbb{N}$, let $\boldsymbol{\mu}_t$ be the kernel of the t^{th}-power endomorphism of $\mathbb{G}_m$. An algebraic group scheme over k is called *linear*, if it is isomorphic to a subgroup scheme of $\mathbf{GL}_n$ for some $n \in \mathbb{N}$.

2.1. Algebraic groups

Let G be a smooth group over k which is of finite type. Let G^0 be the identity component of G. We have a short exact sequence

$$0 \to G^0 \to G \to G/G^0 \to 0, \tag{1}$$

where the quotient group G/G^0 is finite and étale. A classical theorem of Chevalley shows that, if k is either perfect or of characteristic 0, then there exists a short exact sequence

$$0 \to L \to G^0 \to A \to 0, \tag{2}$$

where A is an abelian variety over k and where L is a connected, smooth, linear group over k. In what follows we assume that (2) exists. Let L^{u} be the *unipotent radical* of L. It is the maximal connected, smooth, normal subgroup of L which is *unipotent* (i.e., which over $\bar{k}$ has a composition series whose factors are $\mathbb{G}_a$ groups). We have a short exact sequence

$$0 \to L^{\mathrm{u}} \to L \to R \to 0, \tag{3}$$

where $R := L/L^{\mathrm{u}}$ is a *reductive group* over k (i.e., it is a smooth, connected, linear group over k whose unipotent radical is trivial). By the *k-rank* of R we mean the greatest non-negative integer s such that $\mathbb{G}_m^s$ is a subgroup of R. If the k-rank of R is equal to the $\bar{k}$-rank of $R_{\bar{k}}$, then we say that R is *split*.

Let $Z(R)$ be the (scheme-theoretical) center of R. It is a group scheme of *multiplicative type* (i.e., over $\bar{k}$ it is the extension of a finite product of $\boldsymbol{\mu}_t$ group schemes by a torus $\mathbb{G}_m^n$; here $n \in \mathbb{N} \cup \{0\}$ and $t \in \mathbb{N}$). The quotient group $R^{\mathrm{ad}} := R/Z(R)$ is called the *adjoint group* of R; it is a reductive group over k whose (scheme-theoretical) center is trivial. Let R^{der} be the *derived group* of R; it is the minimal, normal subgroup of R with the property that the quotient group $R^{\mathrm{ab}} := R/R^{\mathrm{der}}$ is abelian. The group R^{ab} is a *torus* (i.e., over $\bar{k}$ it is isomorphic to $\mathbb{G}_m^n$). The groups R^{ad} and R^{der} are *semisimple*. We have two short exact sequences

$$0 \to Z(R) \to R \to R^{\mathrm{ad}} \to 0 \tag{4}$$

and

$$0 \to R^{\mathrm{der}} \to R \to R^{\mathrm{ab}} \to 0. \tag{5}$$

The short exact sequences (1) to (5) are intrinsically associated to G.

If $G = \mathbf{GL}_n$, then $Z(G)$ and G^{ab} are isomorphic to $\mathbb{G}_m$, $G^{\mathrm{der}} = \mathbf{SL}_n$, and $G^{\mathrm{ad}} = \mathbf{PGL}_n$. If $G = \mathbf{GSp}_{2n}$, then $Z(G)$ and G^{ab} are isomorphic to $\mathbb{G}_m$, $G^{\mathrm{der}} = \mathbf{Sp}_{2n}$, and $G^{\mathrm{ad}} = \mathbf{PGSp}_{2n} = \mathbf{Sp}_{2n}/\boldsymbol{\mu}_2$. If $G = \boldsymbol{SO}_{2n+1}$, then $Z(G)$ and G^{ab} are trivial and therefore from (4) and (5) we get that $G = G^{\mathrm{der}} = G^{\mathrm{ad}}$.

2.1.1. Examples of semisimple groups over $\mathbb{Q}$

Let $a, b \in \mathbb{N} \cup \{0\}$ with $a+b > 0$. Let $\mathbf{SU}(a,b)$ be the simply connected semisimple group over $\mathbb{Q}$ whose $\mathbb{Q}$–valued points are the $\mathbb{Q}(i)$–valued points of $\mathbf{SL}_{a+b,\mathbb{Q}}$ that leave invariant the hermitian form $-z_1\overline{z}_1 - \cdots - z_a\overline{z}_a + z_{a+1}\overline{z}_{a+1} + \cdots + z_{a+b}\overline{z}_{a+b}$ over $\mathbb{Q}(i)$. Let $\mathbf{SO}(a,b)$ be the semisimple group over $\mathbb{Q}$ of $a+b$ by $a+b$ matrices of determinant 1 that leave invariant the quadratic form $-x_1^2 - \cdots - x_a^2 + x_{a+1}^2 + \cdots + x_{a+b}^2$ on $\mathbb{Q}^{a+b}$. Let $\mathbf{SO}_a := \mathbf{SO}(0,a)$. Let $\mathbf{SO}^*(2a)$ be the semisimple group over $\mathbb{Q}$ whose group of $\mathbb{Q}$–valued points is the subgroup of $\mathbf{SO}_{2n}(\mathbb{Q}(i))$ that leaves invariant the skew hermitian form $-z_1\overline{z}_{n+1} + z_{n+1}\overline{z}_1 - \cdots - z_n\overline{z}_{2n} + z_{2n}\overline{z}_n$ over $\mathbb{Q}(i)$ (z_i's and x_i's are related here over $\mathbb{Q}(i)$ via $z_i = x_i$).

Definition 1. *By a reductive group scheme $\mathcal{R}$ over a scheme Z, we mean a smooth group scheme over Z which is an affine Z-scheme and whose fibres are reductive groups over fields.*

As above, one defines group schemes $Z(\mathcal{R})$, $\mathcal{R}^{\mathrm{ad}}$, $\mathcal{R}^{\mathrm{der}}$, and $\mathcal{R}^{\mathrm{ab}}$ over Z which are affine Z-schemes. The group scheme $Z(\mathcal{R})$ is of multiplicative type. The group schemes $\mathcal{R}^{\mathrm{ad}}$ and $\mathcal{R}^{\mathrm{der}}$ are semisimple. The group scheme $\mathcal{R}^{\mathrm{ab}}$ is a torus.

2.2. Weil restrictions

Let $i : l \hookrightarrow k$ be a separable finite field extension. Let G be a group scheme over k which is of finite type. Let $\mathrm{Res}_{k/l}$ be the group scheme over l obtained from G through the Weil restriction of scalars. Thus $\mathrm{Res}_{k/l} G$ is defined by the functorial group identification

$$\mathrm{Hom}(Y, \mathrm{Res}_{k/l} G) = \mathrm{Hom}(Y \times_l k, G), \tag{6}$$

where Y is an arbitrary l-scheme. We have

$$(\mathrm{Res}_{k/l} G)_{\bar{k}} = \mathrm{Res}_{k\otimes_l \bar{k}/\bar{k}} G_{k\otimes_l \bar{k}} = \prod_{e\in \mathrm{Hom}_l(k,\bar{k})} G \times_{k,e} \bar{k}. \tag{7}$$

From (7) we easily get that:

(*) if G is a reductive (resp. connected, smooth, affine, linear, unipotent, torus, semisimple, or abelian variety) group over k, then $\mathrm{Res}_{k/l} G$ is a reductive (resp. connected, smooth, affine, linear, unipotent, torus, semisimple, or abelian variety) group over l.

If $j : m \hookrightarrow l$ is another separable finite field extension, then we have a canonical and functorial identification

$$\mathrm{Res}_{l/m} \mathrm{Res}_{k/l} G = \mathrm{Res}_{k/m} G$$

as one can easily check starting from formula (6).

If H is a group scheme over l, then we have a natural closed embedding homomorphism

$$H \hookrightarrow \mathrm{Res}_{k/l} H_k \tag{8}$$

over l which at the level of l-valued points induces the standard monomorphism $H(l) \hookrightarrow H(k) = \mathrm{Res}_{k/l} H_k(l)$.

2.3. Hodge structures

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m$ be the two dimensional torus over $\mathbb{R}$ whose group of $\mathbb{R}$-valued points is the multiplicative group $(\mathbb{C} \setminus \{0\}, \cdot)$ of $\mathbb{C}$. As schemes, we have $\mathbb{S} = \mathrm{Spec}(\mathbb{R}[x, y][\frac{1}{x^2+y^2}])$. By applying (8) we get that we have a short exact sequence

$$0 \to \mathbb{G}_m \to \mathbb{S} \to \mathbf{SO}_{2,\mathbb{R}} \to 0. \tag{9}$$

The group $\mathbf{SO}_{2,\mathbb{R}}(\mathbb{R})$ is isomorphic to the unit circle and thus to $\mathbb{R}/\mathbb{Z}$. The short exact sequence (9) does not split; this is so as $\mathbb{S}$ is isomorphic to $(\mathbb{G}_m \times_{\mathbb{R}} \mathbf{SO}_{2,\mathbb{R}})/\boldsymbol{\mu}_2$, where $\boldsymbol{\mu}_2$ is embedded diagonally into the product.

We have $\mathbb{S}(\mathbb{R}) = \mathbb{C} \setminus \{0\}$. We identify $\mathbb{S}(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C}) = (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$ in such a way that the natural monomorphism $\mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$ induces the map $z \to (z, \overline{z})$, where $z \in \mathbb{C} \setminus \{0\}$.

Let S be a $\mathbb{Z}$-subalgebra of $\mathbb{R}$ (in most applications, we have $S \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$). Let V_S be a free S-module of finite rank. Let $V_{\mathbb{R}} := V_S \otimes_S \mathbb{R}$. By a *Hodge S-structure* on V_S we mean a homomorphism

$$\rho : \mathbb{S} \to \mathbf{GL}_{V_{\mathbb{R}}}. \tag{10}$$

We have a direct sum decomposition

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{(r,s)\in\mathbb{Z}^2} V^{r,s} \tag{11}$$

with the property that $(z_1, z_2) \in \mathbb{S}(\mathbb{C})$ acts via $\rho_{\mathbb{C}}$ on $V^{r,s}$ as the scalar multiplication with $z_1^{-r} z_2^{-s}$. Thus the element $z \in \mathbb{S}(\mathbb{R})$ acts via ρ on $V^{r,s}$ as the scalar multiplication with $z^{-r}\overline{z}^{-s}$. Therefore z acts on $\overline{V^{r,s}}$ as the scalar multiplication with $z^{-s}\overline{z}^{-r}$. This implies that for all $(r, s) \in \mathbb{Z}^2$ we have an identity

$$V^{s,r} = \overline{V^{r,s}}. \tag{12}$$

Conversely, each direct sum decomposition (11) that satisfies the identities (12), is uniquely associated to a homomorphism as in (10).

By the *type* of the Hodge S-structure on V_S, we mean any symmetric subset τ of $\mathbb{Z}^2$ with the property that we have a direct sum decomposition

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{(r,s)\in\tau} V^{r,s}.$$

Here symmetric refers to the fact that if $(r, s) \in \tau$, then we also have $(s, r) \in \tau$. If we can choose τ such that the sum $n := r + s$ does not depend on $(r, s) \in \tau$, one says that the Hodge S-structure on V_S has *weight* n.

2.3.1. Polarizations

For $n \in \mathbb{Z}$, let $S(n)$ be the Hodge S-structure on $(2\pi i)^n S$ which has type $(-n, -n)$. Suppose that the Hodge S-structure on V_S has weight n. By a polarization of the Hodge S-structure on V_S we mean a morphism $\psi : V_S \otimes_S V_S \to S(-n)$ of Hodge S-structures such that the bilinear form $(2\pi i)^n \psi(x \otimes \rho(i)y)$ defined for $x, y \in V_{\mathbb{R}}$, is symmetric and positive definite. Here we identify ψ with its scalar extension to $\mathbb{R}$.

2.3.2. Example

Let A be an abelian variety over $\mathbb{C}$. We take $S = \mathbb{Z}$. Let $V_{\mathbb{Z}} = H^1(A^{\mathrm{an}}, \mathbb{Z})$ be the first cohomology group of the analytic manifold $A^{\mathrm{an}} := A(\mathbb{C})$ with coefficients in $\mathbb{Z}$. Then the classical Hodge theory provides us with a direct sum decomposition

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}, \tag{13a}$$

where $V^{1,0} = H^0(A, \Omega)$ and $V^{0,1} = H^1(A, \mathcal{O}_A)$ (see [Mu, Ch. I, 1]). Here $\mathcal{O}_A$ is the structured ring sheaf on A and Ω is the $\mathcal{O}_A$-module of 1-forms on A. We have $\overline{V^{1,0}} = V^{0,1}$ and therefore (13a) defines a Hodge $\mathbb{Z}$-structure on V_S. Let $F^1(V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) := V^{1,0}$; it is called the *Hodge filtration* of $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

Let $W_{\mathbb{Z}} := \mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z}) = H_1(A^{\mathrm{an}}, \mathbb{Z})$. Let $W_{\mathbb{R}} := W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. Taking the dual of (13a), we get a Hodge $\mathbb{Z}$-structure on $W_{\mathbb{Z}}$ of the form

$$W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = W^{-1,0} \oplus W^{0,-1}. \tag{13b}$$

One can identify naturally $W^{-1,0} = \mathrm{Hom}(V^{1,0}, \mathbb{C}) = \mathrm{Lie}(A)$. Each $z \in \mathbb{S}(\mathbb{R})$ acts on the complex vector space $\mathrm{Lie}(A)$ as the multiplication with z and this explains the convention on negative power signs used in the paragraph after formula (11). We have canonical identifications

$$A^{\mathrm{an}} = W_{\mathbb{Z}} \backslash \mathrm{Lie}(A) = W_{\mathbb{Z}} \backslash (W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) / W^{0,-1}.$$

If λ is a polarization of A, then the non-degenerate form

$$\psi : W_{\mathbb{Z}} \otimes_{\mathbb{Z}} W_{\mathbb{Z}} \to \mathbb{Z}(1) \tag{14}$$

defined naturally by λ, is a polarization of the Hodge $\mathbb{Z}$-structure on $W_{\mathbb{Z}}$.

We have $\mathrm{End}(V_{\mathbb{Z}}) = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} W_{\mathbb{Z}} = \mathrm{End}(W_{\mathbb{Z}})$. Due to the identities (13a) and (13b), the Hodge $\mathbb{Z}$-structure on $\mathrm{End}(V_{\mathbb{Z}})$ is of type

$$\tau_{\mathrm{ab}} := \{(-1, 1), (0, 0), (1, -1)\}. \tag{15}$$

Definition 2. *We use the notations of Example 2.3.1. Let $W := W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. By the Mumford–Tate group of the complex abelian variety A, we mean the smallest subgroup H_A of $\mathbf{GL}_W$ with the property that the homomorphism $x_A : \mathbb{S} \to \mathbf{GL}_{W_{\mathbb{R}}}$ that defines the Hodge $\mathbb{Z}$-structure on $W_{\mathbb{Z}}$, factors through $H_{A,\mathbb{R}}$.*

Proposition 1. *The group H_A is a reductive group over $\mathbb{Q}$.*

Proof: From its very definition, the group H_A is connected. To prove the Proposition it suffices to show that the unipotent radical H_A^{u} of H_A is trivial. Let W_1 be the largest rational subspace of W on which H_A^{u} acts trivially. As H_A^{u} is a normal subgroup of H_A, W_1 is an H_A-module. Thus x_A normalizes $W_1 \otimes_{\mathbb{Q}} \mathbb{R}$ and therefore we have a direct sum decomposition

$$W_1 \otimes_{\mathbb{Q}} \mathbb{C} = [(W_1 \otimes_{\mathbb{Q}} \mathbb{C}) \cap W^{-1,0}] \oplus [(W_1 \otimes_{\mathbb{Q}} \mathbb{C}) \cap W^{0,-1}].$$

Thus $(W_{\mathbb{Z}} \cap W_1)\backslash(W_1 \otimes_{\mathbb{Q}} \mathbb{C})/[(W_1 \otimes_{\mathbb{Q}} \mathbb{C}) \cap W^{0,-1}]$ is a closed analytic submanifold A_1^{an} of A^{an}. A classical theorem of Serre asserts that A_1^{an} is algebraizable i.e., it is the analytic submanifold associated to an abelian subvariety A_1 of A. The short exact sequence $0 \to A_1 \to A \to A/A_1 \to 0$ splits up to isogenies (i.e., A is isogeneous to $A_1 \times_{\mathbb{C}} A_2$, where $A_2 := A/A_1$). Let $W_2 := H_1(A_2^{\mathrm{an}}, \mathbb{Q})$. We have a direct sum decomposition $W = W_1 \oplus W_2$ whose extension to $\mathbb{R}$ is normalized by x_A. Thus the direct sum decomposition $W = W_1 \oplus W_2$ is normalized by H_A. In particular, W_2 is an H_A^{u}-module.

If $W_1 \neq W$, then the unipotent group H_A^{u} acts trivially on a non-zero subspace of W_2 and this represents a contradiction with the largest property of W_1. Thus $W_1 = W$ i.e., H_A^{u} acts trivially on W. Therefore H_A^{u} is the trivial group. $\square$

2.4. Tensors

Let M be a free module of finite rank over a commutative $\mathbb{Z}$-algebra C. Let $M^* := \mathrm{Hom}_C(M, C)$. By the *essential tensor algebra* of $M \oplus M^*$ we mean the C-module

$$\mathcal{T}(M) := \oplus_{s,t \in \mathbb{N} \cup \{0\}} M^{\otimes s} \otimes_C M^{*\otimes t}.$$

Let $F^1(M)$ be a direct summand of M. Let $F^0(M) := M$ and $F^2(M) := 0$. Let $F^1(M^*) := 0$, $F^0(M^*) := \{y \in M^* | y(F^1(M)) = 0\}$, and $F^{-1}(M^*) := M^*$. Let $(F^i(\mathcal{T}(M)))_{i \in \mathbb{Z}}$ be the tensor product filtration of $\mathcal{T}(M)$ defined by the exhaustive, separated filtrations $(F^i(M))_{i \in \{0,1,2\}}$ and $(F^i(M^*))_{i \in \{-1,0,1\}}$ of M and M^* (respectively). We refer to $(F^i(\mathcal{T}(M)))_{i \in \mathbb{Z}}$ as the filtration of $\mathcal{T}(M)$ defined by $F^1(M)$ and to each $F^i(\mathcal{T}(M))$ as the F^i-filtration of $\mathcal{T}(M)$ defined by $F^1(M)$.

We identify naturally $\mathrm{End}(M) = M \otimes_C M^* \subseteq \mathcal{T}(M)$ and $\mathrm{End}(\mathrm{End}(M)) = M^{\otimes 2} \otimes_C M^{*\otimes 2}$. Let $x \in C$ be a non-divisor of 0. A family of tensors of $\mathcal{T}(M[\frac{1}{x}]) = \mathcal{T}(M)[\frac{1}{x}]$ is denoted $(u_\alpha)_{\alpha \in \mathcal{J}}$, with $\mathcal{J}$ as the set of indexes. Let M_1 be another free C-module of finite rank. Let $(u_{1,\alpha})_{\alpha \in \mathcal{J}}$ be a family of tensors of $\mathcal{T}(M_1[\frac{1}{x}])$ indexed also by the set $\mathcal{J}$. By an isomorphism

$$(M, (u_\alpha)_{\alpha \in \mathcal{J}}) \tilde{\to} (M_1, (u_{1,\alpha})_{\alpha \in \mathcal{J}})$$

we mean a C-linear isomorphism $M \tilde{\to} M_1$ that extends naturally to a C-linear isomorphism $\mathcal{T}(M[\frac{1}{x}]) \tilde{\to} \mathcal{T}(M_1[\frac{1}{x}])$ which takes u_α to $u_{1,\alpha}$ for all $\alpha \in \mathcal{J}$. We emphasize that we will denote two tensors or bilinear forms in the same way, provided they are obtained one from another via either a reduction modulo some ideal or a scalar extension.

2.5. Hodge cycles on abelian schemes

We will use the terminology of [De3] on Hodge cycles on an abelian scheme B_X over a reduced $\mathbb{Q}$–scheme X. Thus we write each Hodge cycle v on B_X as a pair $(v_{\mathrm{dR}}, v_{\acute{e}t})$, where v_{dR} and $v_{\acute{e}t}$ are the *de Rham* and the *étale component* of v (respectively). The étale component $v_{\acute{e}t}$ at its turn has an l-component $v_{\acute{e}t}^l$, for each rational prime l.

In what follows we will be interested only in Hodge cycles on B_X that involve no Tate twists and that are tensors of different essential tensor algebras. Accordingly, if X is the spectrum of a field E, then in applications $v^l_{ét}$ will be a suitable $\mathrm{Gal}(\overline{E}/E)$-invariant tensor of $\mathcal{T}(H^1_{ét}(B_{\overline{X}},\mathbb{Q}_l))$, where $\overline{X} := \mathrm{Spec}(\overline{E})$. If $\overline{E}$ is a subfield of $\mathbb{C}$, then we will also use the Betti realization v_B of v. The tensor v_B has the following two properties (that define Hodge cycles on B_X which involve no Tate twist; see [De3, Sect. 2]):

(i) it is a tensor of $\mathcal{T}(H^1((B_X \times_X \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}},\mathbb{Q}))$ that corresponds to v_{dR} (resp. to $v^l_{ét}$) via the canonical isomorphism that relates the Betti cohomology of $(B_X \times_X \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}}$ with $\mathbb{Q}$–coefficients with the de Rham (resp. the $\mathbb{Q}_l$ étale) cohomology of $B_X \times_X \mathrm{Spec}(\mathbb{C})$;

(ii) it is also a tensor of the F^0-filtration of the filtration of $\mathcal{T}(H^1((B_X \times_X \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}},\mathbb{C}))$ defined by the Hodge filtration $F^1(H^1((B_X \times_X \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}},\mathbb{C}))$ of $H^1((B_X \times_X \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}},\mathbb{C})$.

We have the following particular example:

(iii) if $v_B \in \mathrm{End}(H^1((B_X \times_X \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}},\mathbb{Q}))$, then from Riemann theorem we get that v_B is the Betti realization of a $\mathbb{Q}$–endomorphism of $B_X \times_X \mathrm{Spec}(\mathbb{C})$ and therefore the Hodge cycle $(v_{\mathrm{dR}}, v_{ét})$ on B_X is defined uniquely by a $\mathbb{Q}$–endomorphism of B_X.

The class of Hodge cycles is stable under pull backs. In particular, if X is a reduced $\mathbb{Q}$–scheme of finite type, then the pull back of $(v_{\mathrm{dR}}, v_{ét})$ via a complex point $\mathrm{Spec}(\mathbb{C}) \to X$, is a Hodge cycle on the complex abelian variety $B_X \times_X \mathrm{Spec}(\mathbb{C})$.

2.5.1. Example

Let A be an abelian variety over $\mathbb{C}$. Let S be an irreducible, closed subvariety of A. Let n be the codimension of S in A. To S one associates classes $[S]_{\mathrm{dR}} \in H^{2n}_{\mathrm{dR}}(A/\mathbb{C})$, $[S]_l \in H^{2n}_{ét}(A,\mathbb{Q}_l)(n)$, and $[S]_B \in H^{2n}(A^{\mathrm{an}},\mathbb{Q})(n)$. If $[S]_{ét} := ([S]_l)_{l \text{ a prime}}$, then the pair $([S]_{\mathrm{dR}}, [S]_{ét})$ is a Hodge cycle on A which involves Tate twists and whose Betti realization is $[S]_B$. One can identify $H^{2n}_{ét}(A,\mathbb{Q}_l)(n)$ with a $\mathbb{Q}_l$-subspace of $H^1_{ét}(A,\mathbb{Q}_l)^{\otimes n} \otimes_{\mathbb{Q}_l} [(H^1_{ét}(A,\mathbb{Q}_l))^*]^{\otimes n}$ and $H^{2n}_{\mathrm{dR}}(A/\mathbb{C})$ with a $\mathbb{C}$-subspace of $H^1_{\mathrm{dR}}(A/\mathbb{C})^{\otimes n} \otimes_{\mathbb{C}} [(H^1_{\mathrm{dR}}(A/\mathbb{C}))^*]^{\otimes n}$; thus one can naturally view $([S]_{\mathrm{dR}}, [S]_{ét})$ as a Hodge cycle on A which involves no Tate twists. The $\mathbb{Q}$–linear combinations of such cycles $([S]_{\mathrm{dR}}, [S]_{ét})$ are called algebraic cycles on A.

3. Shimura varieties

In this section we introduce Shimura varieties and their basic properties and main types. All continuous actions are in the sense of [De2, Subsubsect. 2.7.1] and are right actions. Thus if a totally discontinuous, locally compact group Γ acts continuously (from the right) on a scheme Y, then for each compact, open subgroup Δ of Γ the geometric quotient scheme Y/Δ exists and the epimorphism $Y \twoheadrightarrow Y/\Delta$ is pro-finite; moreover, we have an identity $Y = \mathrm{proj.lim.}_{\Delta} Y/\Delta$.

3.1. Shimura pairs

A *Shimura pair* $(G, \mathcal{X})$ consists of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class $\mathcal{X}$ of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ that satisfy Deligne's axioms of [De2, Subsubsect. 2.1.1]:

(i) the Hodge $\mathbb{Q}$–structure on $\mathrm{Lie}(G)$ defined by each element $x \in \mathcal{X}$ is of type $\tau_{\mathrm{ab}} = \{(-1, 1), (0, 0), (1, -1)\}$;

(ii) no simple factor of the adjoint group G^{ad} of G becomes compact over $\mathbb{R}$;

(iii) $\mathrm{Ad}(x(i))$ is a Cartan involution of $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}})$, where $\mathrm{Ad} : G_{\mathbb{R}} \to \mathbf{GL}_{\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}})}$ is the adjoint representation.

Axiom (iii) is equivalent to the fact that the adjoint group $G^{\mathrm{ad}}_{\mathbb{R}}$ has a faithful representation $G^{\mathrm{ad}}_{\mathbb{R}} \hookrightarrow \mathbf{GL}_{V_{\mathbb{R}}}$ with the property that there exists a polarization of the Hodge $\mathbb{R}$-structure on $V_{\mathbb{R}}$ defined naturally by any $x \in \mathcal{X}$ which is fixed by $G^{\mathrm{ad}}_{\mathbb{R}}$. These axioms imply that $\mathcal{X}$ has a natural structure of a hermitian symmetric domain, cf. [De2, Cor. 1.1.17].

For $x \in \mathcal{X}$ we consider the *Hodge cocharacter*

$$\mu_x : \mathbb{G}_m \to G_{\mathbb{C}}$$

defined on complex points by the rule: $z \in \mathbb{G}_m(\mathbb{C})$ is mapped to $x_{\mathbb{C}}(z, 1) \in G_{\mathbb{C}}(\mathbb{C})$.

Let $E(G, \mathcal{X}) \hookrightarrow \mathbb{C}$ be the number subfield of $\mathbb{C}$ that is the field of definition of the $G(\mathbb{C})$-conjugacy class $[\mu_{\mathcal{X}}]$ of the cocharacters μ_x's of $G_{\mathbb{C}}$, cf. [Mi2, p. 163]. More precisely $[\mu_{\mathcal{X}}]$ is defined naturally by a $G(\overline{\mathbb{Q}})$-conjugacy class $[\mu^{\overline{\mathbb{Q}}}_{\mathcal{X}}]$ of cocharacters $\mathbb{G}_m \to G_{\overline{\mathbb{Q}}}$; the Galois group $\mathrm{Gal}(\mathbb{Q})$ acts naturally on the set of such $G(\overline{\mathbb{Q}})$-conjugacy classes and $E(G, \mathcal{X})$ is the number field which is the fixed field of the stabilizer subgroup of $[\mu^{\overline{\mathbb{Q}}}_{\mathcal{X}}]$ in $\mathrm{Gal}(\mathbb{Q})$. One calls $E(G, \mathcal{X})$ the *reflex field* of $(G, \mathcal{X})$.

We define the *complex Shimura space*

$$\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}} := \mathrm{proj.lim.}_{K \in \sigma(G)} G(\mathbb{Q}) \backslash (\mathcal{X} \times G(\mathbb{A}_f)/K),$$

where $\sigma(G)$ is the set of compact, open subgroups of $G(\mathbb{A}_f)$ endowed with the inclusion relation (see [De1], [De2], and [Mi1] to [Mi4]). Thus $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}(\mathbb{C})$ is a normal complex space on which $G(\mathbb{A}_f)$ acts. We have an identity

$$\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}(\mathbb{C}) = G(\mathbb{Q}) \backslash [\mathcal{X} \times (G(\mathbb{A}_f)/\overline{Z(G)(\mathbb{Q})})], \tag{16}$$

where $\overline{Z(G)(\mathbb{Q})}$ is the topological closure of $Z(G)(\mathbb{Q})$ in $G(\mathbb{A}_f)$ (cf. [De2, Prop. 2.1.10]). Let $x \in \mathcal{X}$ and $a, g \in G(\mathbb{A}_f)$. Let $[x, a] \in \mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}(\mathbb{C})$ be the point defined naturally by the equivalence class of $(x, a) \in \mathcal{X} \times G(\mathbb{A}_f)$, cf. (16). The $G(\mathbb{A}_f)$-action on $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}(\mathbb{C})$ is defined by the rule $[x, a] \cdot g := [x, ag]$.

For $\ddagger$ a compact subgroup of $G(\mathbb{A}_f)$ let $\mathrm{Sh}_{\ddagger}(G, \mathcal{X})_{\mathbb{C}}(\mathbb{C}) := \mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}(\mathbb{C})/\ddagger$. Let $K \in \sigma(G)$. We can write $\mathrm{Sh}_K(G, \mathcal{X})_{\mathbb{C}}(\mathbb{C}) = G(\mathbb{Q}) \backslash (\mathcal{X} \times G(\mathbb{A}_f)/K)$ as a disjoint union of normal complex spaces of the form $\Sigma \backslash \mathcal{X}^0$, where $\mathcal{X}^0$ is a connected

component of $\mathcal{X}$ and Σ is an *arithmetic subgroup* of $G(\mathbb{Q})$ (i.e., is the intersection of $G(\mathbb{Q})$ with a compact, open subgroup of $G(\mathbb{A}_f)$). A classical result of Baily and Borel allows us to view naturally $\mathrm{Sh}_K(G,\mathcal{X})_{\mathbb{C}}(\mathbb{C}) = G(\mathbb{Q})\backslash(\mathcal{X}\times G(\mathbb{A}_f)/K)$ as the complex space associated to a finite, disjoint union $\mathrm{Sh}_K(G,\mathcal{X})_{\mathbb{C}}$ of normal, quasi-projective, connected varieties over $\mathbb{C}$ (see [BB, Thm. 10.11]). Thus $\mathrm{Sh}_K(G,\mathcal{X})_{\mathbb{C}}$ is a normal, quasi-projective $\mathbb{C}$-scheme and

$$\mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}} := \mathrm{proj.lim.}_{K\in\sigma(G)}\,\mathrm{Sh}_K(G,\mathcal{X})_{\mathbb{C}}$$

is a normal $\mathbb{C}$-scheme on which $G(\mathbb{A}_f)$ acts. We have a canonical identification $\mathrm{Sh}_K(G,\mathcal{X})_{\mathbb{C}} = \mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}}/K$. If K is small enough, then K acts freely on $\mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}}$ and thus $\mathrm{Sh}_K(G,\mathcal{X})_{\mathbb{C}}$ is in fact a smooth, quasi-projective $\mathbb{C}$-scheme.

3.1.1. Example

Let A be an abelian variety over $\mathbb{C}$. Let H_A be its Mumford–Tate group. Let $x_A : \mathbb{S} \to H_{A,\mathbb{R}}$ be the homomorphism that defines the Hodge $\mathbb{Z}$-structure on $W_A := H_1(A^{\mathrm{an}},\mathbb{Z})$, cf. Definition 2. Let $\mathcal{X}_A$ be the $H_A(\mathbb{R})$-conjugacy class of x_A. We check that the pair $(H_A,\mathcal{X}_A)$ is a Shimura pair. The fact that the axiom 3.1 (i) holds for $(H_A,\mathcal{X}_A)$ is implied by (15). If H_A^{ad} has a (non-trivial) simple factor $\diamond$ which over $\mathbb{R}$ is compact, then the fact that $\mathcal{X}_A$ is a hermitian symmetric domain implies that the image of x_A in $\diamond_{\mathbb{R}}$ is trivial and this contradicts the smallest property (see Definition 2) of the Mumford–Tate group H_A. Thus the axioms 3.1 (ii) holds for $(H_A,\mathcal{X}_A)$. The fact that the axioms 3.1 (iii) holds is implied by the fact that B has a polarization and thus by the fact that (14) holds. We emphasize that the reflex field $E(H_A,\mathcal{X}_A)$ can be any CM number field.

3.1.2. Example

The most studied Shimura pairs are constructed as follows. Let W be a vector space over $\mathbb{Q}$ of even dimension $2r$. Let ψ be a non-degenerate alternative form on W. Let $\mathcal{S}$ be the set of all monomorphisms $\mathbb{S}\hookrightarrow \mathbf{GSp}(W\otimes_{\mathbb{Q}}\mathbb{R},\psi)$ that define Hodge $\mathbb{Q}$–structures on W of type $\{(-1,0),(0,-1)\}$ and that have either $2\pi i\psi$ or $-2\pi i\psi$ as polarizations. Thus $\mathcal{S}$ is two copies of the Siegel domain of genus r (the two copies correspond to either $2\pi i\psi$ or $-2\pi i\psi$ being a polarization of the resulting Hodge $\mathbb{Q}$–structures on W). It is easy to see that $\mathcal{S}$ is a $\mathbf{GSp}(W,\psi)(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S}\to\mathbf{GSp}(W\otimes_{\mathbb{Q}}\mathbb{R},\psi)$. One can choose an abelian variety A over $\mathbb{C}$ such that in fact we have $(\mathbf{GSp}(W,\psi),\mathcal{S}) = (H_A,\mathcal{X}_A)$ and therefore $(\mathbf{GSp}(W,\psi),\mathcal{S})$ is a Shimura pair, cf. Example 3.1.1. We call $(\mathbf{GSp}(W,\psi),\mathcal{S})$ a Shimura pair that defines a Siegel modular variety $\mathrm{Sh}(\mathbf{GSp}(W,\psi),\mathcal{S})$ (to be defined in Subsection 3.2 below). As $\mathbf{GSp}(W,\psi)$ is a split group, the $\mathbf{GSp}(W,\psi)(\overline{\mathbb{Q}})$-conjugacy class $[\mu_{\mathcal{X}}^{\overline{\mathbb{Q}}}]$ is defined naturally by a cocharacter of $\mathbf{GSp}(W,\psi)$ and therefore we have $E(\mathbf{GSp}(W,\psi),\mathcal{S}) = \mathbb{Q}$.

3.1.3. Example

Let n be a positive integer. Let $G := \mathbf{SO}(2,n)$; it is the identity component of the group that fixes the quadratic from $-x_1^2 - x_2^2 + x_3^2 + \cdots + x_{n+2}^2$ on $\mathbb{Q}^{n+2}$. The group G has a subgroup $\mathbf{SO}_2\times_{\mathbb{Q}}\mathbf{SO}_n$ which normalizes the rational vector

subspaces of $\mathbb{Q}^{n+2}$ generated by the first two and by the last n vectors of the standard $\mathbb{Q}$–basis for $\mathbb{Q}^{n+2}$. Let $x : \mathbb{S} \to G_{\mathbb{R}}$ be a homomorphism whose image is the subgroup $\mathbf{SO}_{2,\mathbb{R}}$ of $G_{\mathbb{R}}$ and whose kernel is the split torus $\mathbb{G}_m$ of $\mathbb{S}$. Let $\mathcal{X}$ be the $G(\mathbb{R})$-conjugacy class of x. Then the pair $(G, \mathcal{X})$ is a Shimura pair.

The group $G_{\mathbb{Q}(i)}$ is split (i.e., $\mathbb{G}_m^{[\frac{n}{2}]}$ is a subgroup of it) and thus the $G(\overline{\mathbb{Q}})$-conjugacy class $[\mu_{\mathcal{X}}^{\overline{\mathbb{Q}}}]$ is defined naturally by a cocharacter $\mu_0 : \mathbb{G}_m \to G_{\mathbb{Q}(i)}$. We can choose μ_0 such that the non-trivial element of $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ takes μ_0 under Galois conjugation to μ_0^{-1}. It is easy to see that the two cocharacters μ_0 and μ_0^{-1} are $G(\mathbb{Q}(i))$-conjugate. Therefore $E(G, \mathcal{X}) = \mathbb{Q}$.

If $n = 19$, then $(G, \mathcal{X})$ is the Shimura pair associated to moduli spaces of polarized K3 surfaces.

3.1.4. Example

Let T be a torus over $\mathbb{Q}$. Let $x : \mathbb{S} \to T_{\mathbb{R}}$ be an arbitrary homomorphism. Then the pair $(T, \{x\})$ is a Shimura pair. Its reflex field $E := E(T, \{x\})$ is the field of definition of the cocharacter $\mu_x : \mathbb{G}_m \to T_{\mathbb{C}}$. We denote also by $\mu_x : \mathbb{G}_m \to T_E$ the homomorphism whose extension to $\mathbb{C}$ is μ_x.

From the homomorphism $\mu_x : \mathbb{G}_m \to T_E$ we get naturally a new one

$$N_x : \mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_m \overset{\mathrm{Res}_{E/\mathbb{Q}}(\mu_x)}{\longrightarrow} \mathrm{Res}_{E/\mathbb{Q}}\, T_E \overset{\mathrm{Norm}\, E/\mathbb{Q}}{\longrightarrow} T.$$

Thus for each commutative $\mathbb{Q}$–algebra C we get a homomorphism $N_x(C) : \mathbb{G}_m(E \otimes_{\mathbb{Q}} C) \to T(C)$.

Let E^{ab} be the maximal abelian extension of E. The reciprocity map

$$r(T, \{x\}) : \mathrm{Gal}(E^{\mathrm{ab}}/E) \to T(\mathbb{A}_f)/\overline{T(\mathbb{Q})}$$

is defined as follows: if $\tau \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$ and if $s \in \mathbb{J}_E$ is an idèle (of E) such that $\mathrm{rec}_E(s) = \tau$, then $r(T, \{x\})(\tau) := N_x(\mathbb{A}_f)(s_f)$, where s_f is the finite part of s. Here the Artin reciprocity map rec_E is such that a uniformizing parameter is mapped to the geometric Frobenius element.

Definition 3. *By a map $f : (G_1, \mathcal{X}_1) \to (G_2, \mathcal{X}_2)$ of Shimura pairs we mean a homomorphism $f : G_1 \to G_2$ of groups over $\mathbb{Q}$ such that for each $x \in \mathcal{X}_1$ we have $f(x) := f_{\mathbb{R}} \circ x \in \mathcal{X}_2$. If $f : G_1 \to G_2$ is a monomorphism, then we say $f : (G_1, \mathcal{X}_1) \to (G_2, \mathcal{X}_2)$ is an injective map. If G_1 is a torus and if $f : (G_1, \mathcal{X}_1) \hookrightarrow (G_2, \mathcal{X}_2)$ is an injective map, then f is called a special pair in $(G_2, \mathcal{X}_2)$.*

3.2. Canonical models

By a *model* of $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}$ over a subfield k of $\mathbb{C}$, we mean a scheme S over k endowed with a continuous right action of $G(\mathbb{A}_f)$ (defined over k), such that there exists a $G(\mathbb{A}_f)$-equivariant isomorphism

$$\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}} \tilde{\to} S_{\mathbb{C}}.$$

The *canonical model* of $\mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}}$ (or of $(G,\mathcal{X})$ itself) is the model $\mathrm{Sh}(G,\mathcal{X})$ of $\mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}}$ over $E(G,\mathcal{X})$ which satisfies the following property:

(*) *if $(T,\{x\})$ is a special pair in $(G,\mathcal{X})$, then for each element $a \in G(\mathbb{A}_f)$ the point $[x,a]$ of $\mathrm{Sh}(G,\mathcal{X})(\mathbb{C}) = \mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}}(\mathbb{C})$ is rational over $E(T,\{x\})^{\mathrm{ab}}$ and every element τ of $\mathrm{Gal}(E(T,\{x\})^{\mathrm{ab}}/E(T,\{x\}))$ acts on $[x,a]$ according to the rule*

$$\tau[x,a] = [x, ar(\tau)],$$

where $r := r(T,\{x\})$ is as in Example 3.1.4.

The canonical model of $\mathrm{Sh}(G,\mathcal{X})$ exists and is uniquely determined by the property (*) up to a unique isomorphism (see [De1], [De2], [Mi2], and [Mi4]).

By the *dimension* d of $\mathrm{Sh}(G,\mathcal{X})$ (or $(G,\mathcal{X})$ or $\mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}}$) we mean the dimension of $\mathcal{X}$ as a complex manifold. One computes d as follows. For $x \in \mathcal{X}$, let $\mathrm{Lie}(G_{\mathbb{C}}) = F_x^{-1,0} \oplus F_x^{0,0} \oplus F_x^{0,-1}$ be the Hodge decomposition defined by x. Let K_∞ be the centralizer of x in $G_{\mathbb{R}}$; it is a reductive group over $\mathbb{R}$ (cf. [Bo, Ch. IV, 13.17, Cor. 2]). We have $\mathrm{Lie}(K_\infty) \otimes_{\mathbb{R}} \mathbb{C} = F_x^{0,0}$ and (as analytic real manifolds) $\mathcal{X} = [G(\mathbb{R})]/[K_\infty(\mathbb{R})]$. Thus as $\dim_{\mathbb{C}}(F^{-1,0}) = \dim_{\mathbb{C}}(F^{0,-1})$, we get that

(17)

$$d = \frac{1}{2}\dim(G_{\mathbb{R}}/K_\infty) = \frac{1}{2}\dim_{\mathbb{C}}(\mathrm{Lie}(G_{\mathbb{C}})/F_x^{0,0}) = \dim_{\mathbb{C}}(F_x^{-1,0}) = \dim_{\mathbb{C}}(F_x^{0,-1}).$$

For $\ddagger$ a compact subgroup of $G(\mathbb{A}_f)$ let $\mathrm{Sh}_{\ddagger}(G,\mathcal{X}) := \mathrm{Sh}(G,\mathcal{X})/\ddagger$. If $K \in \sigma(G)$, then $\mathrm{Sh}_K(G,\mathcal{X})$ is a normal, quasi-projective $E(G,\mathcal{X})$-scheme which is equidimensional of dimension d and whose extension to $\mathbb{C}$ is (canonically identified with) the $\mathbb{C}$-scheme $\mathrm{Sh}_K(G,\mathcal{X})_{\mathbb{C}}$ we have introduced in Subsection 3.1.

If $f : (G_1,\mathcal{X}_1) \to (G_2,\mathcal{X}_2)$ is a map between two Shimura pairs, then $E(G_2,\mathcal{X}_2)$ is a subfield of $E(G_1,\mathcal{X}_1)$ and there exists a unique $G_1(\mathbb{A}_f)$-equivariant morphism (still denoted by f)

$$f : \mathrm{Sh}(G_1,\mathcal{X}_1) \to \mathrm{Sh}(G_2,\mathcal{X}_2)_{E(G_1,\mathcal{X}_1)} \tag{18}$$

which at the level of complex points is the map $[x,a] \to [f(x), f(a)]$ ([De1, Cor. 5.4]). We get as well a $G(\mathbb{A}_f)$-equivariant morphism (denoted in the same way)

$$f : \mathrm{Sh}(G_1,\mathcal{X}_1) \to \mathrm{Sh}(G_2,\mathcal{X}_2)$$

of $E(G_2,\mathcal{X}_2)$-schemes. If f is an injective map, then based on (16) one gets that (18) is in fact a closed embedding.

3.3. Classification of Shimura pairs

Let $(G,\mathcal{X})$ be a Shimura pair. If $x \in \mathcal{X}$, let $x^{\mathrm{ab}} : \mathbb{S} \to G_{\mathbb{R}}^{\mathrm{ab}}$ and $x^{\mathrm{ad}} : \mathbb{S} \to G_{\mathbb{R}}^{\mathrm{ad}}$ be the homomorphisms defined naturally by $x : \mathbb{S} \to G_{\mathbb{R}}$. The homomorphism x^{ab} does not depend on $x \in \mathcal{X}$ and the Shimura pair $(G^{\mathrm{ab}}, \{x^{\mathrm{ab}}\})$ has dimension 0. Let $\mathcal{X}^{\mathrm{ad}}$ be the $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class of x^{ad}. The Shimura pairs $(G^{\mathrm{ab}}, \{x^{\mathrm{ab}}\})$ and $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}})$ are called the *toric* and the *adjoint* (respectively) Shimura pairs of

$(G, \mathcal{X})$. The centralizer $K_{\infty,\mathrm{ad}}$ of x^{ad} in $G^{\mathrm{ad}}_{\mathbb{R}}$ is a reductive group over $\mathbb{R}$ which is a maximal compact subgroup of $G^{\mathrm{ad}}_{\mathbb{R}}$. The hermitian symmetric domain structure on $\mathcal{X}^{\mathrm{ad}}$ is obtained via the natural identification $\mathcal{X}^{\mathrm{ad}} = [G^{\mathrm{ad}}(\mathbb{R})]/[K_{\infty,\mathrm{ad}}(\mathbb{R})]$. The hermitian symmetric domain $\mathcal{X}$ is a finite union of connected components of $\mathcal{X}^{\mathrm{ad}}$. In particular, we have $\mathcal{X} \subseteq \mathcal{X}^{\mathrm{ad}}$.

We have a product decomposition

$$(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}}) = \prod_{i \in I} (G_i, \mathcal{X}_i) \tag{19}$$

into *simple adjoint* Shimura pairs, where each G_i is a simple group over $\mathbb{Q}$. For each $i \in I$ there exists a number field F_i such that we have an isomorphism $G_i \tilde{\to} \mathrm{Res}_{F_i/\mathbb{Q}} G_i^{F_i}$, where $G_i^{F_i}$ is an absolutely simple adjoint group over F_i (see [Ti, Subsubsect. 3.1.2]). The number field F_i is uniquely determined up to $\mathrm{Gal}(\mathbb{Q})$-conjugation (i.e., up to isomorphism).

Axiom 3.2 (iii) is equivalent to the fact that $G^{\mathrm{ad}}_{\mathbb{R}}$ is an inner form of its compact form $G^{\mathrm{ad,c}}_{\mathbb{R}}$, cf. [De2, p. 255]. Thus $G^{\mathrm{ad}}_{\mathbb{R}}$ is a product of absolutely simple, adjoint groups over $\mathbb{R}$. But for each $i \in I$ we have $G_{i,\mathbb{R}} \tilde{\to} \mathrm{Res}_{F_i \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{R}} [G_i^{F_i} \times_{F_i} (F_i \otimes_{\mathbb{Q}} \mathbb{R})]$. From the last two sentences, we get that for each $i \in I$ the $\mathbb{R}$-algebra $F_i \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to a finite number of copies of $\mathbb{R}$. In other words, for each $i \in I$ the number field F_i is totally real.

We have the following conclusions of the last three paragraphs:

(i) Let G be a reductive group over $\mathbb{Q}$. To give a Shimura pair $(G, \mathcal{X})$ is the same thing as to give a Shimura pair $(G^{\mathrm{ab}}, \{x^{\mathrm{ab}}\})$ of dimension 0 (i.e., a homomorphism $x^{\mathrm{ab}} : \mathbb{S} \to G^{\mathrm{ab}}_{\mathbb{R}}$) and an adjoint Shimura pair $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}})$, with the properties that for an (any) element $x^{\mathrm{ad}} \in \mathcal{X}^{\mathrm{ad}}$ the homomorphism $(x^{\mathrm{ab}}, x^{\mathrm{ad}}) : \mathbb{S} \to G^{\mathrm{ab}}_{\mathbb{R}} \times_{\mathbb{R}} G^{\mathrm{ad}}_{\mathbb{R}}$ lifts to a homomorphism $x : \mathbb{S} \to G_{\mathbb{R}}$, where $G_{\mathbb{R}} \to G^{\mathrm{ab}}_{\mathbb{R}} \times_{\mathbb{R}} G^{\mathrm{ad}}_{\mathbb{R}}$ is the standard isogeny. One takes $\mathcal{X}$ to be the $G(\mathbb{R})$-conjugacy class of x. We emphasize that the Shimura pair $(G, \mathcal{X})$ can depend on the choice of $x^{\mathrm{ad}} \in \mathcal{X}^{\mathrm{ad}}$ (though its isomorphism class does not).

(ii) To give an adjoint Shimura pair $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}})$ is the same thing as to give a finite set $(G_i, \mathcal{X}_i)$ of simple adjoint Shimura pairs, cf. (19).

(iii) To give a simple adjoint Shimura pair $(G_i, \mathcal{X}_i)$, one has to first give a totally real number field F_i and an absolutely simple, adjoint group $G_i^{F_i}$ over F_i that satisfies the following property:

(*) for each embedding $j : F_i \hookrightarrow \mathbb{R}$, the group $G_i^{F_i} \times_{F_i,j} \mathbb{R}$ is either compact (and then one defines $\mathcal{X}_{i,j}$ to be a set with one element) or is not compact and associated naturally to a connected hermitian symmetric domain $\mathcal{X}_{i,j}$.

The product $\prod_{j \in \mathrm{Hom}(F_i,\mathbb{R})} \mathcal{X}_{i,j}$ is a connected hermitian symmetric domain isomorphic to the connected components of $\mathcal{X}_i$. If $G^{F_i}_{i,\mathbb{C}}$ is of classical Lie type and if $G_i^{F_i} \times_{F_i,j} \mathbb{R}$ is not compact, then $G_i^{F_i} \times_{F_i,j} \mathbb{R}$ is isomorphic to either $\mathbf{SU}(a,b)^{\mathrm{ad}}_{\mathbb{R}}$ with $a, b \geq 1$, or $\mathbf{SO}(2,n)^{\mathrm{ad}}_{\mathbb{R}}$ with $n \geq 1$, or $\mathbf{Sp}^{\mathrm{ad}}_{2n,\mathbb{R}}$ with $n \geq 1$, or $\mathbf{SO}^*(2n)^{\mathrm{ad}}_{\mathbb{R}}$

with $n \geq 4$. The last think one has to give is a family of homomorphisms $x_{i,j} : \mathbb{S}/\mathbb{G}_m \to G_i^{F_i} \times_{F_i,j} \mathbb{R}$, where

- $x_{i,j}$ is trivial if $G_i^{F_i} \times_{F_i,j} \mathbb{R}$ is compact, and
- $x_{i,j}$ identifies $\mathbb{S}/\mathbb{G}_m = \mathbf{SO}_{2,\mathbb{R}}$ with the identity component of the center of a maximal compact subgroup of $G_i^{F_i} \times_{F_i,j} \mathbb{R}$ if $G_i^{F_i} \times_{F_i,j} \mathbb{R}$ is not compact.

One takes $\mathcal{X}_i$ to be the $G_i(\mathbb{R})$-conjugacy class of the composite of the natural epimorphism $\mathbb{S} \twoheadrightarrow \mathbb{S}/\mathbb{G}_m$ with $\prod_{j \in \mathrm{Hom}(F_i,\mathbb{R})} x_{i,j} : \mathbb{S}/\mathbb{G}_m \to G_{i,\mathbb{R}}$. Once F_i and $G_i^{F_i}$ are given, there exist a finite number of possibilities for $\mathcal{X}_i$ (they correspond to possible replacements of some of the $x_{i,j}$'s by their inverses).

3.3.1. Shimura types

A Shimura variety $\mathrm{Sh}(G_1, \mathcal{X}_1)$ is called *unitary* if the adjoint group G_1^{ad} is non-trivial and all simple factors of $G_{1,\mathbb{C}}^{\mathrm{ad}}$ are **PGL** groups over $\mathbb{C}$.

Let $(G, \mathcal{X})$ be a simple, adjoint Shimura pair. Let $\mathfrak{L}$ be the Lie type of anyone of the simple factors of $G_{\mathbb{C}}$. If $\mathfrak{L}$ is either A_n, B_n, C_n, E_6, or E_7, then one say that $(G, \mathcal{X})$ is of $\mathfrak{L}$ Shimura type. If $\mathfrak{L}$ is D_n with $n \geq 4$, then there exist three disjoint possibilities for the type of $(G, \mathcal{X})$: they are $D_n^{\mathbb{H}}$, $D_n^{\mathbb{R}}$, and D_n^{mixed}. If $n \geq 5$, then $(G, \mathcal{X})$ is of $D_n^{\mathbb{H}}$ (resp. of $D_n^{\mathbb{R}}$) Shimura type if and only if each simple, non-compact factor of $G_{\mathbb{R}}$ is isomorphic to $\mathbf{SO}^*(2n)_{\mathbb{R}}^{\mathrm{ad}}$ (resp. to $\mathbf{SO}(2, 2n-2)_{\mathbb{R}}^{\mathrm{ad}}$). The only if part of the previous sentence holds even if $n = 4$.

We will not detail here the precise difference between the Shimura types $D_4^{\mathbb{H}}$, $D_4^{\mathbb{R}}$, and D_4^{mixed} (see [De2, p. 272]).

3.4. Shimura varieties of Hodge type

Let $(G, \mathcal{X})$ be a Shimura pair. We say that $\mathrm{Sh}(G, \mathcal{X})$ (or $(G, \mathcal{X})$) is of *Hodge type*, if there exists an injective map $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{S})$ into a Shimura pair that defines a Siegel modular variety. The Hodge $\mathbb{Q}$–structure on W defined by any $x \in \mathcal{X}$ is of type $\{(-1, 0), (0, -1)\}$, cf. (13b). This implies that $x(\mathbb{G}_m)$ is the group of scalar automorphisms of $\mathbf{GL}_{W \otimes_{\mathbb{Q}} \mathbb{R}}$. Therefore $Z(G)$ contains the group $\mathbb{G}_m = Z(\mathbf{GL}_W)$ of scalar automorphisms of W. The image of $Z(G)_{\mathbb{R}}$ in $\mathbf{GSp}(W, \psi)_{\mathbb{R}}^{\mathrm{ad}}$ is contained in the centralizer of the image of x in $\mathbf{GSp}(W, \psi)_{\mathbb{R}}^{\mathrm{ad}}$ and thus it is contained in a compact group. From the last two sentences we get that we have a short exact sequence

$$0 \to \mathbb{G}_m \to Z(G) \to Z(G)^{\mathrm{c}} \to 0, \tag{20}$$

where $Z(G)_{\mathbb{R}}^{\mathrm{c}}$ is a compact group of multiplicative type. In this way we get the only if part of the following Proposition (see [De2, Prop. 2.3.2 or Cor. 2.3.4]).

Proposition 2. *A Shimura pair $(G, \mathcal{X})$ is of Hodge type if and only if the following two properties hold:*

(i) *there exists a faithful representation $G \hookrightarrow \mathbf{GL}_W$ with the property that the Hodge $\mathbb{Q}$–structure on W defined by a (any) $x \in \mathcal{X}$ is of type $\{(-1, 0), (0, -1)\}$;*

(ii) *we have a short exact sequence as in (20).*

If $(G, \mathcal{X})$ is of Hodge type, then (16) becomes (cf. [De2, Cor. 2.1.11])

$$\mathrm{Sh}(G, \mathcal{X})(\mathbb{C}) = G(\mathbb{Q}) \backslash (\mathcal{X} \times G(\mathbb{A}_f)).$$

3.4.1. Moduli interpretation

Let $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{S})$ be an injective map. We fix a family $(s_\alpha)_{\alpha \in \mathcal{J}}$ of tensors of $\mathcal{T}(W^*)$ such that G is the subgroup of $\mathbf{GSp}(W, \psi)$ that fixes s_α for all $\alpha \in \mathcal{J}$, cf. [De3, Prop. 3.1 (c)]. Let L be a $\mathbb{Z}$-lattice of W such that we have a perfect form $\psi : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$. We follow [Va1, Subsect. 4.1] to present the standard moduli interpretation of the complex Shimura variety $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}$ with respect to the $\mathbb{Z}$-lattice L of W and the family of tensors $(s_\alpha)_{\alpha \in \mathcal{J}}$.

We consider quadruples of the form $[A, \lambda_A, (v_\alpha)_{\alpha \in \mathcal{J}}, k]$ where:

(a) (A, λ_A) is a principally polarized abelian variety over $\mathbb{C}$;

(b) $(v_\alpha)_{\alpha \in \mathcal{J}}$ is a family of Hodge cycles on A;

(c) k is an isomorphism $H_1(A^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \tilde{\to} L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ whose tensorization with $\mathbb{Q}$ (denoted also by k) takes the Betti realization of v_α into s_α for all $\alpha \in \mathcal{J}$ and which induces a symplectic similitude isomorphism between $(H_1(A^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}, \lambda_A)$ and $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}, \psi)$.

We define $\mathcal{A}(G, \mathcal{X}, W, \psi)$ to be the set of isomorphism classes of quadruples of the above form that satisfy the following two conditions:

(i) there exists a similitude isomorphism $(H_1(A^{\mathrm{an}}, \mathbb{Q}), \lambda_A) \tilde{\to} (W, \psi)$ that takes the Betti realization of v_α into s_α for all $\alpha \in \mathcal{J}$;

(ii) by composing the homomorphism $x_A : \mathbb{S} \to \mathbf{GSp}(H_1(A^{\mathrm{an}}, \mathbb{R}), \lambda_A)$ that defines the Hodge $\mathbb{R}$-structure on $H_1(A^{\mathrm{an}}, \mathbb{R})$ with an isomorphism of real groups $\mathbf{GSp}(H_1(A^{\mathrm{an}}, \mathbb{R}), \lambda_A) \tilde{\to} \mathbf{GSp}(W \otimes_{\mathbb{Q}} \mathbb{R}, \psi)$ induced naturally by an isomorphism as in (i), we get an element of $\mathcal{X}$.

We have a right action of $G(\mathbb{A}_f)$ on $\mathcal{A}(G, \mathcal{X}, W, \psi)$ defined by the rule:

$$[A, \lambda_A, (v_\alpha)_{\alpha \in \mathcal{J}}, k] \cdot g := [A', \lambda_{A'}, (v_\alpha)_{\alpha \in \mathcal{J}}, g^{-1}k].$$

Here A' is the abelian variety which is isogeneous to A and which is defined by the $\mathbb{Z}$-lattice $H_1(A'^{,\mathrm{an}}, \mathbb{Z})$ of $H_1(A'^{,\mathrm{an}}, \mathbb{Q}) = H_1(A^{\mathrm{an}}, \mathbb{Q})$ whose tensorization with $\widehat{\mathbb{Z}}$ is $(k^{-1} \circ g)(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$, while $\lambda_{A'}$ is the only rational multiple of λ_A which produces a principal polarization of A' (see [De1, Thm. 4.7] for the theorem of Riemann used here). Here as well as in (e) below, we will identify a polarization with its Betti realization.

There exists a $G(\mathbb{A}_f)$-equivariant bijection

$$f_{(G, \mathcal{X}, W, \psi)} : \mathrm{Sh}(G, \mathcal{X})(\mathbb{C}) \tilde{\to} \mathcal{A}(G, \mathcal{X}, W, \psi)$$

defined as follows. To $[x, g] \in \mathrm{Sh}(G, \mathcal{X})(\mathbb{C}) = G(\mathbb{Q}) \setminus (\mathcal{X} \times G(\mathbb{A}_f))$ we associate the quadruple $f_{(G, \mathcal{X}, W, \psi)}([x, g]) := [A, \lambda_A, (v_\alpha)_{\alpha \in \mathcal{J}}, k]$ where:

(d) A is associated to the Hodge $\mathbb{Q}$–structure on W defined by x and to the unique $\mathbb{Z}$-lattice $H_1(A^{\mathrm{an}}, \mathbb{Z})$ of $H_1(A^{\mathrm{an}}, \mathbb{Q}) = W$ for which we have an isomorphism $k = g^{-1} : H_1(A^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \tilde{\to} L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ induced naturally by the automorphism g^{-1} of $W \otimes_{\mathbb{Q}} \mathbb{A}_f$; thus we have $A^{\mathrm{an}} = H_1(A^{\mathrm{an}}, \mathbb{Z}) \backslash (W \otimes_{\mathbb{Q}} \mathbb{C})/W_x^{0,-1}$, where $W \otimes_{\mathbb{Q}} \mathbb{C} = W_x^{-1,0} \oplus W_x^{0,-1}$ is the Hodge decomposition defined by x;

(e) λ_A is the only rational multiple of ψ which gives birth to a principal polarization of A;

(f) for each $\alpha \in \mathcal{J}$, the Betti realization of v_α is s_α.

The inverse $g_{(G,\mathcal{X},W,\psi)}$ of $f_{(G,\mathcal{X},W,\psi)}$ is defined as follows. We consider a quadruple $[A, \lambda_A, (v_\alpha)_{\alpha \in \mathcal{J}}, k] \in \mathcal{A}(G, \mathcal{X}, W, \psi)$. We choose a symplectic similitude isomorphism $i_A : (H_1(A^{\mathrm{an}}, \mathbb{Q}), \lambda_A) \tilde{\to} (W, \psi)$ as in (i). It gives birth naturally to an isomorphism $\tilde{i}_A : \mathbf{GSp}(H_1(A^{\mathrm{an}}, \mathbb{Q}), \lambda_A) \to \mathbf{GSp}(W, \psi)$ of groups over $\mathbb{Q}$. We define $x \in \mathcal{X}$ to be $\tilde{i}_{A,\mathbb{R}} \circ x_A$ (with x_A as in Definition 2) and $g \in G(\mathbb{A}_f)$ to be the composite isomorphism $W \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{k^{-1}} H_1(A^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{i_A \otimes 1_{\mathbb{A}_f}} W \otimes_{\mathbb{Q}} \mathbb{A}_f$. Then

$$g_{(G,\mathcal{X},W,\psi)}([A, \lambda_A, (v_\alpha)_{\alpha \in \mathcal{J}}, k]) := [x, g].$$

Taking $(G, \mathcal{X}) = (\mathbf{GSp}(W, \psi), \mathcal{S})$ and $\mathcal{J} = \emptyset$, we get a bijection between the set $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})(\mathbb{C})$ and the set of isomorphism classes of principally polarized abelian varieties over $\mathbb{C}$ of dimension $\frac{1}{2}\dim_{\mathbb{Q}}(W)$ that have (compatibly) level-N symplectic similitude structures for all positive integers N. Thus to give a $\mathbb{C}$-valued point of $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})$ is the same thing as to give a triple $[A, \lambda_A, (l_N)_{N \in \mathbb{N}}]$, where (A, λ_A) is a principally polarized abelian variety over $\mathbb{C}$ of dimension $\frac{1}{2}\dim_{\mathbb{Q}}(W)$ and where $l_N : (L/NL, \psi) \tilde{\to} (H_1(A^{\mathrm{an}}, \mathbb{Z}/N\mathbb{Z}), \lambda_A)$'s are forming a compatible system of symplectic similitude isomorphisms. The compatibility means here that if N_1 and N_2 are positive integers such that $N_1 | N_2$, then l_{N_1} is obtained from l_{N_2} by tensoring with $\mathbb{Z}/N_1\mathbb{Z}$.

3.4.2. Canonical models

Let $r := \frac{1}{2}\dim_{\mathbb{Q}}(W) \in \mathbb{N}$. Let $N \geq 3$ be a positive integer. Let $\mathcal{A}_{r,1,N}$ be the Mumford–moduli scheme over $\mathbb{Z}[\frac{1}{N}]$ that parametrizes isomorphism classes of principally polarized abelian schemes over $\mathbb{Z}[\frac{1}{N}]$-schemes that have level-N symplectic similitude structure and that have relative dimension r, cf. [MFK, Thms. 7.9 and 7.10]. We consider the $\mathbb{Q}$–scheme

$$\mathcal{A}_{r,1,\mathrm{all}} := \mathrm{proj.lim.}_{N \in \mathbb{N}} \mathcal{A}_{r,1,N}.$$

We have a natural identification $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})(\mathbb{C}) = \mathcal{A}_{r,1,\mathrm{all}}(\mathbb{C})$ of sets, cf. end of Subsubsection 3.4.1. One can easily check that this identification is in fact an isomorphism of complex manifolds. From the very definition of the algebraic structure on $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})_{\mathbb{C}}$ (obtained based on [BB, Thm. 10.11]), one gets that there exists a natural identification $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})_{\mathbb{C}} = \mathcal{A}_{r,1,\mathrm{all},\mathbb{C}}$ of $\mathbb{C}$-schemes. Classical works of Shimura, Taniyama, etc., show that the last identification is the extension to $\mathbb{C}$ of an identification $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S}) = \mathcal{A}_{r,1,\mathrm{all}}$ of $\mathbb{Q}$–schemes.

The reflex field $E(G,\mathcal{X})$ is the *smallest* number field such that the closed subscheme $\mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}}$ of $\mathrm{Sh}(\mathbf{GSp}(W,\psi),\mathcal{S})_{\mathbb{C}}$ is defined over $E(G,\mathcal{X})$. In other words, we have a natural closed embedding (cf. end of Subsection 3.2)

$$f:\mathrm{Sh}(G,\mathcal{X})\hookrightarrow \mathcal{A}_{r,1,\mathrm{all},E(G,\mathcal{X})}=\mathrm{Sh}(\mathbf{GSp}(W,\psi),\mathcal{S})_{E(G,\mathcal{X})}. \tag{21}$$

The pull back $(\mathcal{V},\Lambda_{\mathcal{V}})$ to $\mathrm{Sh}(G,\mathcal{X})$ of the universal principally polarized abelian scheme over $\mathcal{A}_{r,1,\mathrm{all},E(G,\mathcal{X})}$ is such that there exists naturally a family of Hodge cycles $(v_\alpha^{\mathcal{V}})_{\alpha\in\mathcal{J}}$ on the abelian scheme $\mathcal{V}$.

If $y:=[x,g]\in\mathrm{Sh}(G,\mathcal{X})(\mathbb{C})$ and if $f_{(G,\mathcal{X},W,\psi)}([x,g])=[A,\lambda_A,(v_\alpha)_{\alpha\in\mathcal{J}},k]$, then each $y^*(v_\alpha^{\mathcal{V}})$ is the Hodge cycle v_α on $A=y^*(\mathcal{V})$.

Definition 4. *Let $(G_1,\mathcal{X}_1)$ be a Shimura pair. We say that $(G_1,\mathcal{X}_1)$ is of preabelian type, if there exists a Shimura pair $(G,\mathcal{X})$ of Hodge type such that we have an isomorphism $(G^{\mathrm{ad}},\mathcal{X}^{\mathrm{ad}})\tilde{\to}(G_1^{\mathrm{ad}},\mathcal{X}_1^{\mathrm{ad}})$ of adjoint Shimura pairs. If moreover this isomorphism $(G^{\mathrm{ad}},\mathcal{X}^{\mathrm{ad}})\tilde{\to}(G_1^{\mathrm{ad}},\mathcal{X}_1^{\mathrm{ad}})$ is induced naturally by an isogeny $G^{\mathrm{der}}\to G_1^{\mathrm{der}}$, then we say that $(G_1,\mathcal{X}_1)$ is of abelian type.*

Remark 1. *Let $(G_1,\mathcal{X}_1)$ be an arbitrary Shimura variety. Let $\rho_1:G_1\hookrightarrow\mathbf{GL}_{W_1}$ be a faithful representation. As in Subsubsection 3.4.1, one checks that $\mathrm{Sh}(G_1,\mathcal{X}_1)_{\mathbb{C}}$ is a moduli space of Hodge $\mathbb{Q}$–structures on W_1 equipped with extra structures. If moreover $(G_1,\mathcal{X}_1)$ is of abelian type, then $\mathrm{Sh}(G_1,\mathcal{X}_1)_{\mathbb{C}}$ is in fact a moduli scheme of polarized abelian motives endowed with Hodge cycles and certain compatible systems of level structures (cf. [Mi3]).*

3.4.3. Classification

Let $(G_1,\mathcal{X}_1)$ be a simple, adjoint Shimura pair. Then $(G_1,\mathcal{X}_1)$ is of abelian type if and only if $(G_1,\mathcal{X}_1)$ is of A_n, B_n, C_n, $D_n^{\mathbb{H}}$, or $D_n^{\mathbb{R}}$ Shimura type. For this classical result due to Satake and Deligne we refer to [Sa1], [Sa2, Part III], and [De2, Table 2.3.8]. There exists a Shimura pair $(G,\mathcal{X})$ of Hodge type whose adjoint is isomorphic to $(G_1,\mathcal{X}_1)$ and whose derived group G^{der} is simply connected if and only if $(G_1,\mathcal{X}_1)$ is of A_n, B_n, C_n, or $D_n^{\mathbb{R}}$ Shimura type (cf. [De2, Table 2.3.8]).

4. Integral models

In this Section we follow [Mi2] and [Va1] to define different integral models of Shimura varieties. Let $p\in\mathbb{N}$ be a prime. Let $\mathbb{Z}_{(p)}$ be the location of $\mathbb{Z}$ at its prime ideal (p). Let $\mathbb{A}_f^{(p)}$ be the ring of finite adèles with the p-component omitted; we have $\mathbb{A}_f=\mathbb{Q}_p\times\mathbb{A}_f^{(p)}$. Let $(G,\mathcal{X})$ be a Shimura pair. Let v be a prime of $E(G,\mathcal{X})$ that divides p. Let $O_{(v)}$ be the local ring of v.

4.1. Basic definitions

(a) Let H be a compact, open subgroup of $G(\mathbb{Q}_p)$. By an *integral model* of $\mathrm{Sh}_H(G,\mathcal{X})$ over $O_{(v)}$ we mean a faithfully flat scheme $\mathcal{N}$ over $O_{(v)}$ together with a $G(\mathbb{A}_f^{(p)})$-continuous action on it and a $G(\mathbb{A}_f^{(p)})$-equivariant isomorphism

$$\mathcal{N}_{E(G,\mathcal{X})} \tilde{\to} \mathrm{Sh}_H(G,\mathcal{X}).$$

When the $G(\mathbb{A}_f^{(p)})$-action on $\mathcal{N}$ is obvious, by abuse of language, we say that the $O_{(v)}$-scheme $\mathcal{N}$ is an integral model. The integral model $\mathcal{N}$ is said to be *smooth* (resp. *normal*) if there exists a compact, open subgroup H_0 of $G(\mathbb{A}_f^{(p)})$ such that for every inclusion $H_2 \subseteq H_1$ of compact, open subgroups of H_0, the natural morphism $\mathcal{N}/H_2 \to \mathcal{N}/H_1$ is a finite étale morphism between smooth schemes (resp. between normal schemes) of finite type over $O_{(v)}$. In other words, there exists a compact open subgroup H_0 of $G(\mathbb{A}_f^{(p)})$ such that $\mathcal{N}$ is a pro-étale cover of the smooth (resp. the normal) scheme $\mathcal{N}/H_0$ of finite type over $O_{(v)}$.

(b) A regular, faithfully flat $O_{(v)}$-scheme Y is called *p-healthy* (resp. *healthy*) regular, if for each open subscheme U of Y which contains $Y_{\mathbb{Q}}$ and all points of Y of codimension 1, every p-divisible group (resp. every abelian scheme) over U extends uniquely to a p-divisible group (resp. extends to an abelian scheme) over Y.

(c) A scheme Z over $O_{(v)}$ is said to have the *extension property* if for each healthy regular scheme Y over $O_{(v)}$, every $E(G,\mathcal{X})$-morphism $Y_{E(G,\mathcal{X})} \to Z_{E(G,\mathcal{X})}$ extends uniquely to an $O_{(v)}$-morphism $Y \to Z$.

(d) A smooth integral model of $\mathrm{Sh}_H(G,\mathcal{X})$ over $O_{(v)}$ that has the extension property is called an *integral canonical model* of $\mathrm{Sh}(G,\mathcal{X})/H$ over $O_{(v)}$.

(e) Let D be a Dedekind domain. Let K be the field of fractions of D. Let Z_K be a smooth scheme of finite type over K. By a *Néron model* of Z_K over D we mean a smooth scheme of finite type Z over D whose generic fibre is Z_K and which is uniquely determined by the following universal property: for each smooth scheme Y over D, every K-morphism $Y_K \to Z_K$ extends uniquely to a D-morphism $Y \to Z$.

(f) The group $G_{\mathbb{Q}_p}$ is called *unramified* if and only if extends to a reductive group scheme $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$. In such a case, each compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ of the form $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ is called a *hyperspecial* subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$.

(g) Let Z be a flat $O_{(v)}$-scheme and let Y be a closed subscheme of $Z_{k(v)}$. The *dilatation* W of Z centered on Y is an affine Z-scheme defined as follows. To define W, we can work locally in the Zariski topology of Z and therefore we van assume that $Z = \mathrm{Spec}(C)$ is an affine scheme. Let I be the ideal of C that defines Y and let π_v be a uniformizer of $O_{(v)}$. Then W is the spectrum of the C-subalgebra of $C[\frac{1}{\pi_v}]$ generated by $\frac{i}{\pi_v}$ with $i \in I$. The affine morphism $W \to Z$ of $O_{(v)}$-schemes enjoys the following universal property. Let $q : \tilde{Z} \to Z$ be a morphism of flat $O_{(v)}$-schemes. Then q factors uniquely through a morphism $\tilde{Z} \to W$ of Z-schemes if and only if $q_{k(v)} : \tilde{Z}_{k(v)} \to Z_{k(v)}$ factors through Y (i.e., $q_{k(v)}$ is a composite morphism $\tilde{Z}_{k(v)} \to Y \hookrightarrow Z_{k(v)}$).

4.2. Classical example

Let $f : (G,\mathcal{X}) \hookrightarrow (\mathbf{GSp}(W,\psi),\mathcal{S})$ be an injective map. Let L be a $\mathbb{Z}$-lattice of W such that ψ induces a perfect form $\psi : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$. Let $N \geq 3$ be a natural

number which is prime to p. Let

$$K(N) := \{g \in \mathbf{GSp}(L,\psi)(\widehat{\mathbb{Z}})|g \bmod N \text{ is identity}\} \text{ and } K_p := \mathbf{GSp}(L,\psi)(\mathbb{Z}_p).$$

We have an identity $K_p = K(N) \cap \mathbf{GSp}(W,\psi)(\mathbb{Q}_p)$. Let

$$\mathcal{M} := \text{proj.lim.}_{N\in\mathbb{N}, g.c.d.(N,p)=1}\mathcal{A}_{r,1,N};$$

it is a $\mathbb{Z}_{(p)}$-scheme that parametrizes isomorphism classes of principally polarized abelian schemes over $\mathbb{Z}_{(p)}$-schemes that have compatible level-N symplectic similitude structures for all $N \in \mathbb{N}$ prime to p and that have relative dimension r.

The totally discontinuous, locally compact group $\mathbf{GSp}(W,\psi)(\mathbb{A}_f^{(p)})$ acts continuously on $\mathcal{M}$ and moreover $\mathcal{M}$ is a pro-étale cover of $\mathcal{A}_{r,1,N,\mathbb{Z}_{(p)}}$ for all $N \in \mathbb{N}$ prime to p. From (21) we get that we can identify $\mathrm{Sh}_{K(N)}(\mathbf{GSp}(W,\psi),\mathcal{S}) = \mathcal{A}_{r,1,N,\mathbb{Q}}$ and $\mathrm{Sh}_{K_p}(\mathbf{GSp}(W,\psi),\mathcal{S}) = \mathcal{M}_{\mathbb{Q}}$. From the last two sentences we get that $\mathcal{M}$ is a smooth integral model of $\mathrm{Sh}_{K_p}(\mathbf{GSp}(W,\psi),\mathcal{S})$ over $\mathbb{Z}_{(p)}$.

Let $G_{\mathbb{Z}_{(p)}}$ be the Zariski closure of G in $\mathbf{GL}_{L\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}}$; it is an affine, flat group scheme over $\mathbb{Z}_{(p)}$ whose generic fibre is G. Let $H(N) := K(N) \cap G(\mathbb{A}_f)$ and $H_p := H(N) \cap G(\mathbb{Q}_p)$. From (21) we easily get that we have finite morphisms

$$f(N) : \mathrm{Sh}_{H(N)}(G,\mathcal{X}) \to \mathrm{Sh}_{K(N)}(\mathbf{GSp}(W,\psi),\mathcal{S}) \tag{22a}$$

and

$$f_p : \mathrm{Sh}_{H_p}(G,\mathcal{X}) \to \mathrm{Sh}_{K_p}(\mathbf{GSp}(W,\psi),\mathcal{S}). \tag{22b}$$

As $N \geq 3$, a principally polarized abelian scheme with level-N structure has no automorphism (see [Mu, Ch. IV, 21, Thm. 5] for this result of Serre). This implies that $K(N)$ acts freely on $\mathcal{A}_{r,1,\mathrm{all},E(G,\mathcal{X})} = \mathrm{Sh}(\mathbf{GSp}(W,\psi),\mathcal{S})_{E(G,\mathcal{X})}$. From this and (21) we get that $H(N)$ acts freely on $\mathrm{Sh}(G,\mathcal{X})$. Therefore the $E(G,\mathcal{X})$-scheme $\mathrm{Sh}_{H(N)}(G,\mathcal{X})$ is smooth and thus $\mathrm{Sh}_{H_p}(G,\mathcal{X})$ is a regular scheme which is formally smooth over $E(G,\mathcal{X})$.

Let $\mathcal{N}(N)$ be the normalization of $\mathcal{A}_{r,1,N}$ in the ring of fractions of $\mathrm{Sh}_{H(N)}(G,\mathcal{X})$ and let $\mathcal{N}_p$ be the normalization of $\mathcal{M}$ in the ring of fractions of $\mathrm{Sh}_{H_p}(G,\mathcal{X})$. [Comment: the role of the integral model $\mathcal{N}$ used in Section 1, will be played in what follows by $\mathcal{N}(N)$.] Let $O(G,\mathcal{X})$ be the ring of integers of $E(G,\mathcal{X})$. Let $O(G,\mathcal{X})_{(p)} := O(G,\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$; it is the normalization of $\mathbb{Z}_{(p)}$ in $E(G,\mathcal{X})$. The scheme $\mathcal{N}(N)$ is a faithfully flat $O(G,\mathcal{X})[\frac{1}{N}]$-scheme which is normal and of finite type and whose generic fibre is $\mathrm{Sh}_{H(N)}(G,\mathcal{X})$ (the finite type part is implied by the fact that $O_{(v)}$ is an excellent ring). The scheme $\mathcal{N}_p$ is a faithfully flat $O(G,\mathcal{X})_{(p)}$-scheme which is normal and whose generic fibre is $\mathrm{Sh}_{H_p}(G,\mathcal{X})$.

One gets the existence of a finite map

$$f(N) : \mathcal{N}(N) \to \mathcal{A}_{r,1,N}$$

and of a pro-finite map

$$f_p : \mathcal{N}_p \to \mathcal{M}$$

that extends naturally (22a) and (22b) (respectively). Moreover, the totally discontinuous, locally compact group $G(\mathbb{A}_f^{(p)})$ acts continuously on $\mathcal{N}_p$. Let

$$\mathcal{N}_v := \mathcal{N}_p \otimes_{O(G,\mathcal{X})_{(p)}} O_{(v)}.$$

Proposition 3. **(a)** *The $O_{(v)}$-scheme $\mathcal{N}_v$ is a normal integral model of* $\mathrm{Sh}(G,\mathcal{X})$ *over $O_{(v)}$. Moreover, $\mathcal{N}_v$ is a pro-étale cover of $\mathcal{N}(N)_{O_{(v)}}$.*

(b) *The morphism $f_p : \mathcal{N}_p \to \mathcal{M}$ is finite.*

Proof: Let H_0 be a compact, open subgroup of $G(\mathbb{A}_f^{(p)})$ such that $H_p \times H_0$ is a compact, open subgroup of $H(N)$. As $H(N)$ acts freely on $\mathcal{M}$, it also acts freely on $\mathcal{N}_p$. This implies that $\mathcal{N}_v$ is a pro-étale cover of both $\mathcal{N}(N)_{O_{(v)}}$ and $\mathcal{N}_v/H_0$. Therefore for all open subgroups H_1 and H_2 of H_0 with $H_1 \leqslant H_2$, the morphism $\mathcal{N}_v/H_1 \to \mathcal{N}_v/H_2$ is a finite morphism between étale covers of $\mathcal{N}(N)_{O_{(v)}}$ and therefore it is an étale cover. Based on this, one easily checks that the right action of $G(\mathbb{A}_f^{(p)})$ on $\mathcal{N}_v$ is continuous. Thus (a) holds.

Part (b) is an easy consequence of the fact that $\mathcal{N}_p/H_0$ is a finite scheme over $\mathcal{M}_{O(G,\mathcal{X})_{(p)}}/H_0$. □

4.2.1. PEL type Shimura varieties

Let $\mathcal{B}$ be the $\mathbb{Q}$–subalgebra of $\mathrm{End}(W)$ formed by elements fixed by G. We consider two axioms:

(*) the group G is the identity component of the centralizer of $\mathcal{B}$ in $\mathbf{GSp}(W,\psi)$;

(**) the group G is the centralizer of $\mathcal{B}$ in $\mathbf{GSp}(W,\psi)$.

If the axiom (*) holds, then one calls $\mathrm{Sh}(G,\mathcal{X})$ a Shimura variety of *PEL type*. If the axiom (**) holds, then one calls $\mathrm{Sh}(G,\mathcal{X})$ a Shimura variety of PEL type of either *A or C type*. Here PEL stands for polarizations, endomorphisms, and level structures while the A and C types refer to the fact that all simple factors of $G^{\mathrm{ad}}_{\mathbb{C}}$ are (under the axiom (**)) of some A_n or C_n Lie type (and not of D_n Lie type with $n \geq 4$).

If the axiom (**) holds, then we can choose the family $(v_\alpha)_{\alpha\in\mathcal{J}}$ to be exactly the family of all elements of $\mathcal{B}$. In such a case, all Hodge cycles mentioned in Subsection 3.4.2 are defined by endomorphisms. Let $\mathcal{B}_{(p)} := \mathcal{B} \cap \mathrm{End}(L_{\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}})$; it is a $\mathbb{Z}_{(p)}$-order of $\mathcal{B}$.

4.2.2. Example

Suppose that $\mathcal{B}_{(p)}$ is a semisimple $\mathbb{Z}_{(p)}$-algebra and that $G_{\mathbb{Z}_{(p)}}$ is the centralizer of $\mathcal{B}_{(p)}$ in the group scheme $\mathbf{GSp}(L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \psi)$. Then $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme and moreover $\mathcal{N}_p$ (resp. $\mathcal{N}_v$) is a moduli scheme of principally polarized abelian schemes which are over $O(G,\mathcal{X})_{(p)}$-schemes (resp. over $O_{(v)}$-schemes), which have relative dimension r, which have compatible level-N symplectic simil-

itude structures for all $N \in \mathbb{N}$ prime to p, which are endowed with a $\mathbb{Z}_{(p)}$-algebra $\mathcal{B}_{(p)}$ of $\mathbb{Z}_{(p)}$-endomorphisms, and which satisfy certain axioms that are related to the properties (d) to (f) of Subsubsection 3.4.2. Unfortunately, presently this is the *only case* when $\mathcal{N}_p$ (resp. when $\mathcal{N}_v$) has a good moduli interpretation. This explains the difficulties one encounters in getting as well as of stating results pertaining to either $\mathcal{N}_p$ or $\mathcal{N}_v$.

4.3. Main problems

Here is a list of six main problems in the study of $\mathcal{N}(N)$, $\mathcal{N}_p$, and $\mathcal{N}_v$. For simplicity, these problems will be stated here only in terms of $\mathcal{N}(N)$.

(a) Determine when $\mathcal{N}(N)$ is uniquely determined up to isomorphism by its generic fibre $\mathrm{Sh}_{H(N)}(G, \mathcal{X})$ and by a suitable universal property.

(b) Determine when $\mathcal{N}(N)$ is a smooth $O(G, \mathcal{X})[\frac{1}{N}]$-scheme.

(c) Determine when $\mathcal{N}(N)$ is a projective $O(G, \mathcal{X})[\frac{1}{N}]$-scheme.

(d) Identify and study different stratifications of the special fibres of $\mathcal{N}(N)$.

(e) Describe the points of $\mathcal{N}(N)$ with values in finite fields.

(f) Describe the points of $\mathcal{N}(N)$ with values in $O[\frac{1}{N}]$, where O is the ring of integers of some finite field extension of $E(G, \mathcal{X})$.

In the next four Sections we will study the first four problems one by one, in a way that could be useful towards the partial solutions of the problem (e). Any approach to the problem (f) would require a very good understanding of the first five problems and this is the reason (as well as the main motivation) for why the six problems are listed together.

5. Uniqueness of integral models

Until the end we will use the following notations introduced in Section 4:

$$f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{S}), \;\; L, \;\; N, \;\; K(N), \;\; K_p, \;\; H(N), \;\; H_p, \;\; O(G, \mathcal{X}),$$

$$O(G, \mathcal{X})_{(p)}, \;\; v, \;\; O_{(v)}, \;\; f(N) : \mathcal{N}(N) \to \mathcal{A}_{r,1,N}, \;\; f_p : \mathcal{N}_p \to \mathcal{M}, \;\; \mathcal{N}_v, \;\; G_{\mathbb{Z}_{(p)}}.$$

Let e_v be the index of ramification of v. Let $k(v)$ be the residue field of v. Let

$$\mathcal{L}(N)_v := \mathcal{N}(N) \otimes_{O(G,\mathcal{X})[\frac{1}{N}]} k(v) \quad \text{and} \quad \mathcal{L}_v := \mathcal{N}_v \otimes_{O_{(v)}} k(v).$$

In this Section we study when the $k(v)$-scheme $\mathcal{L}(N)_v$ (resp. $\mathcal{L}_v$) is uniquely determined in some sensible way by $\mathrm{Sh}_{H(N)}(G, \mathcal{X})$ (resp. by $\mathrm{Sh}_{H_p}(G, \mathcal{X})$) and by the prime v of $E(G, \mathcal{X})$. Whenever one gets such a uniqueness property, one

can call $\mathcal{L}(N)_v$ (resp. $\mathcal{L}_v$) as the *canonical fibre model* of $\mathrm{Sh}_{H(N)}(G,\mathcal{X})$ (resp. $\mathrm{Sh}_{H_p}(G,\mathcal{X})$) at v (or over $k(v)$).

Milne's original insight (see [Mi2] and [Va1]) was to prove in many cases the uniqueness of $\mathcal{N}_v$ and $\mathcal{L}_v$ by showing first that:

(i) the $O_{(v)}$-scheme $\mathcal{N}_v$ has the extension property, and

(ii) $\mathcal{N}_v$ is a healthy regular scheme in the sense of Definition 4.1 (b).

While (i) always holds (see Proposition 4 below), it is very hard in general to decide if (ii) holds. However, results of [Va1], [Va2], and [Va11] allow us to get that (ii) holds in many cases of interest (see Subsection 5.1). Subsection 5.2 shows how one gets the uniqueness of $\mathcal{N}(N)$ (and therefore also of $\mathcal{L}(N)_v$) via (the uniqueness of) Néron models.

Proposition 4. *The $O_{(v)}$-scheme $\mathcal{N}_v$ has the extension property.*

Proof: Let Y be a healthy regular scheme over $O_{(v)}$. Let $q : Y_{E(G,\mathcal{X})} \to \mathrm{Sh}_{H_p}(G,\mathcal{X})$ be a morphism of $E(G,\mathcal{X})$-schemes. Let $(\mathcal{U},\lambda_{\mathcal{U}})$ be the pull back to $Y_{E(G,\mathcal{X})}$ of the universal principally polarized abelian scheme over $\mathcal{M}$ (via the composite morphism $Y_{E(G,\mathcal{X})} \to \mathrm{Sh}_{H_p}(G,\mathcal{X}) \to \mathrm{Sh}_{K_p}(\mathbf{GSp}(W,\psi),\mathcal{S}) = \mathcal{M}_{\mathbb{Q}}$). As the universal principally polarized abelian scheme over $\mathcal{M}$ has a level-N symplectic similitude structure for all $N \in \mathbb{N}$ prime to p, the same holds for $(\mathcal{U},\lambda_{\mathcal{U}})$. From this and the Néron–Ogg–Shafarevich criterion of good reduction (see [BLR, Ch. 7, 7.4, Thm. 5]) we get that $\mathcal{U}$ extends to an abelian scheme $\mathcal{U}_U$ over an open subscheme U of Y which contains $Y_{\mathbb{Q}} = Y_{E(G,\mathcal{X})}$ and for which we have $\mathrm{codim}_Y(Y \setminus U) \geq 2$. Thus $\mathcal{U}_U$ extends to an abelian scheme $\mathcal{U}_Y$ over Y, cf. the very definition of a healthy regular scheme. The polarization $\lambda_{\mathcal{U}}$ extends as well to a polarization $\lambda_{\mathcal{U}_Y}$ of $\mathcal{U}_Y$, cf. [Mi2, Prop. 2.14]. Moreover, each level-N symplectic similitude structure of $(\mathcal{U},\lambda_{\mathcal{U}})$ extends to a level-N symplectic similitude structure of $(\mathcal{U}_Y,\lambda_{\mathcal{U}_Y})$. This implies that the composite of q with the finite morphism $\mathrm{Sh}_{H_p}(G,\mathcal{X}) \to \mathcal{M}_{E(G,\mathcal{X})}$ extends uniquely to a morphism $Y \to \mathcal{M}_{O_{(v)}}$. As Y is a regular scheme and thus also normal and as $\mathcal{N}_v \to \mathcal{M}_{O_{(v)}}$ is a finite morphism, we get that the morphism $Y \to \mathcal{M}_{O_{(v)}}$ factors uniquely through $\mathcal{N}_v$ (as it does so generically). This implies that $q : Y_{E(G,\mathcal{X})} \to \mathrm{Sh}_{H_p}(G,\mathcal{X})$ extends uniquely to a morphism $q_Y : Y \to \mathcal{N}_v$ of $O_{(v)}$-schemes. From this the Proposition follows. $\square$

Proposition 5. *Let Y be a regular scheme which is faithfully flat over $\mathbb{Z}_{(p)}$. Then the following two properties hold:*

(a) *Let U be an open subscheme of Y which contains $Y_{\mathbb{Q}}$ and the generic points of $Y_{\mathbb{F}_p}$. Let A_U be an abelian scheme over U with the property that its p-divisible group D_U extends to a p-divisible group D over Y. Then A_U extends to an abelian scheme A over U.*

(b) *If Y is a p-healthy regular scheme, then it is also a healthy regular scheme.*

Proof: Part (b) follows from (a) and the very definitions. To prove (a) we follow [Va2, Prop. 4.1]. Let $N \geq 3$ be a positive integer relatively prime to p.

To show that A exists, we can assume that Y is local, complete, and strictly henselian, that U is the complement of the maximal point y of Y, that A_U has a principal polarization λ_{A_U}, and that (A_U, λ_{A_U}) has a level-N symplectic similitude structure $l_{U,N}$ (see [FC, (i)-(iii) of pp. 185, 186]). We write $Y = \mathrm{Spec}(R)$. Let λ_{D_U} be the principal quasi-polarization of D_U defined naturally by λ_{A_U}; it extends to a principal quasi-polarization λ_D of D (cf. Tate's theorem [Ta, Thm. 4]). Let r be the relative dimension of A_U. Let $(\mathcal{A}, \Lambda_{\mathcal{A}})$ be the universal principally polarized abelian scheme over $\mathcal{A}_{r,1,N}$.

Let $m_U : U \to \mathcal{A}_{r,1,N}$ be the morphism defined by $(A_U, \lambda_{A_U}, l_{U,N})$. We show that m_U extends to a morphism $m : Y \to \mathcal{A}_{r,1,N}$.

Let $N_0 \in \mathbb{N}$ be prime to p. From the classical purity theorem we get that the étale cover $A_U[N_0] \to U$ extends to an étale cover $Y_{N_0} \to Y$. But as Y is strictly henselian, Y has no connected étale cover different from Y. Thus each Y_{N_0} is a disjoint union of N_0^{2r}-copies of Y. From this we get that (A_U, λ_{A_U}) has a level-N_0 symplectic similitude structure l_{U,N_0} for every $N_0 \in \mathbb{N}$ prime to p.

Let $\overline{\mathcal{A}}_{r,1,N}$ be a projective, toroidal compactification of $\mathcal{A}_{r,1,N}$ such that (cf. [FC, Chap. IV, Thm. 6.7]):

(a) the complement of $\mathcal{A}_{r,1,N}$ in $\overline{\mathcal{A}}_{r,1,N}$ has pure codimension 1 in $\overline{\mathcal{A}}_{r,1,N}$ and

(b) there exists a semi-abelian scheme over $\overline{\mathcal{A}}_{r,1,N}$ that extends $\mathcal{A}$.

Let $\tilde{Y}$ be the normalization of the Zariski closure of U in $Y \times_{\mathbb{Z}} \overline{\mathcal{A}}_{r,1,N}$. It is a projective, normal, integral Y-scheme which has U as an open subscheme. Let C be the complement of U in $\tilde{Y}$ endowed with the reduced structure; it is a reduced, projective scheme over the residue field k of y. The $\mathbb{Z}$-algebras of global functions of Y, U, and $\tilde{Y}$ are all equal to R (cf. [Ma, Thm. 38] for U). Thus C is a connected k-scheme, cf. [Ha, Ch. III, Cor. 11.3] applied to $\tilde{Y} \to Y$.

Let $\overline{A}_{\tilde{Y}}$ be the semi-abelian scheme over $\tilde{Y}$ that extends A_U (it is unique, cf. [FC, Chap. I, Prop. 2.7]). Due to the existence of the l_{U,N_0}'s, the Néron–Ogg–Shafarevich criterion implies that $\overline{A}_{\tilde{Y}}$ is an abelian scheme in codimension at most 1. Therefore, since the complement of $\mathcal{A}_{r,1,N}$ in $\overline{\mathcal{A}}_{r,1,N}$ has pure codimension 1 in $\overline{\mathcal{A}}_{r,1,N}$, it follows that $\overline{A}_{\tilde{Y}}$ is an abelian scheme. Thus m_U extends to a morphism $m_{\tilde{Y}} : \tilde{Y} \to \mathcal{A}_{r,1,N}$. Let $\lambda_{\overline{A}_{\tilde{Y}}} := m_{\tilde{Y}}^*(\Lambda_{\mathcal{A}})$. Tate's theorem implies that the principally quasi-polarized p-divisible group of $(\overline{A}_{\tilde{Y}}, \lambda_{\overline{A}_{\tilde{Y}}})$ is the pull-back $(D_{\tilde{Y}}, \lambda_{D_{\tilde{Y}}})$ of (D, λ_D) to $\tilde{Y}$. Hence the pull back (D_C, λ_{D_C}) of $(D_{\tilde{Y}}, \lambda_{D_{\tilde{Y}}})$ to C is constant i.e., it is the pull back to C of a principally quasi-polarized p-divisible group over k.

We check that the image $m_{\tilde{Y}}(C)$ of C through $m_{\tilde{Y}}$ is a point $\{y_0\}$ of $\mathcal{A}_{r,1,N}$. Since C is connected, to check this it suffices to show that, if $\widehat{O_c}$ is the completion of the local ring O_c of C at an arbitrary point c of C, then the morphism $\mathrm{Spec}(\widehat{O_c}) \to \mathcal{A}_{r,1,N}$ defined naturally by $m_{\tilde{Y}}$ is constant. But as (D_C, λ_{D_C}) is constant, this follows from Serre–Tate deformation theory (see [Me, Chaps. 4, 5]). Thus $m_{\tilde{Y}}(C)$ is a point $\{y_0\}$ of $\mathcal{A}_{r,1,N}$.

Let R_0 be the local ring of $\mathcal{A}_{r,1,N}$ at y_0. Because Y is local and $\tilde{Y}$ is a projective Y-scheme, each point of $\tilde{Y}$ specializes to a point of C. Hence each point of the image of $m_{\tilde{Y}}$ specializes to y_0 and thus $m_{\tilde{Y}}$ factors through the natural mor-

phism $\mathrm{Spec}(R_0) \to \mathcal{A}_{r,1,N}$. Since R is the ring of global functions of $\tilde{Y}$, the resulting morphism $\tilde{Y} \to \mathrm{Spec}(R_0)$ factors through a morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(R_0)$. Therefore $m_{\tilde{Y}}$ factors through a morphism $m : Y \to \mathcal{A}_{r,1,N}$ that extends m_U. This ends the argument for the existence of m. We conclude that $A := m^*(\mathcal{A})$ is an abelian scheme over Y which extends A_U. Thus (a) holds. $\square$

5.1. Examples of healthy regular schemes

In (the proofs of) [FC, Ch. IV, Thms. 6.4, 6.4', and 6.8] was claimed that every regular scheme which is faithfully flat over $\mathbb{Z}_{(p)}$ is p-healthy regular as well as healthy regular. In turns out that this claim is far from being true. For instance, an example of Raynaud–Gabber–Ogus (see [dJO1, Sect. 6]) shows that the regular scheme $\mathrm{Spec}(W(k)[[x,y]]/((xy)^{p-1}-p))$ is neither p-healthy nor healthy regular. Here $W(k)$ is the ring of Witt vectors with coefficients in a perfect field k of characteristic p.

Based on Proposition 5 (b) and a theorem of Raynaud (see [Ra, Thm. 3.3.3]), one easily checks that if $e_v < p-2$, then each regular scheme which is formally smooth over $O_{(v)}$ is a healthy regular scheme (see [Va1, Subsubsect. 3.2.17]). In [Va2, Thm. 1.3] it is proved that the same holds provided $e_v = 1$. In [Va11] it is proved that the same holds provided $e_v = p-1$. Even more, in [Va11, Thm. 1.3 and Cor. 1.5] it is proved that:

Theorem 1. **(a)** *Suppose that $p > 2$. Each regular scheme which is formally smooth over $O_{(v)}$ is healthy regular if and only if the following inequality holds $e_v \leq p-1$.*

(b) *Suppose that $p = 2$ and $e_v = 1$. Then each regular scheme which is formally smooth over $O_{(v)}$ is healthy regular.*

Part (a) also holds for $p = 2$ but this is not checked loc. cit. and this is why above for $p = 2$ we stated only one implication in the form of (b). From Theorem 1 and Propositions 3 (a) and 4 we get the following answer to the problem 4.3 (a):

Corollary 1. *Suppose that $e_v \leq p-1$ and that $\mathcal{N}_v$ is a regular scheme which is formally smooth over $O_{(v)}$ (i.e., and that $\mathcal{N}(N)_{O_{(v)}}$ is a smooth $O_{(v)}$-scheme). Then $\mathcal{N}_v$ is the integral canonical model of $\mathrm{Sh}_{H_p}(G,\mathcal{X})$ over $O_{(v)}$ and it is uniquely determined up to unique isomorphism. Thus also $\mathcal{L}(N)_v$ and $\mathcal{L}_v$ are uniquely determined by $\mathrm{Sh}_{H(N)}(G,\mathcal{X})$ and $\mathrm{Sh}_{H_p}(G,\mathcal{X})$ (respectively) and v.*

5.1.1. Example

The integral canonical model of $\mathrm{Sh}_{K_p}(\mathbf{GSp}(W,\psi),\mathcal{S})$ over $\mathbb{Z}_{(p)}$ is $\mathcal{M}$.

5.2. Integral models as Nèron models

In [Ne] it is showed that each abelian variety over the field of fractions K of a Dedekind domain D has a Néron model over D. In [BLR] it is checked that many other closed subschemes of torsors of certain commutative group varieties over K, have Néron models over D. But most often, for $N >> 0$ the $E(G,\mathcal{X})$-scheme $\mathrm{Sh}_{H(N)}(G,\mathcal{X})$ can not be embedded in such torsors; we include one basic example.

5.2.1. Example

Suppose that $G^{\mathrm{ad}}_{\mathbb{R}}$ is isomorphic to $\mathbf{SU}(a,b)^{\mathrm{ad}}_{\mathbb{R}} \times_{\mathbb{R}} \mathbf{SU}(a+b,0)^{\mathrm{ad}}_{\mathbb{R}}$ for some positive integers $a \geq 3$ and $b \geq 3$. One has $H^{1,0}(\mathcal{C}(\mathbb{C}),\mathbb{C}) = 0$ for each connected component $\mathcal{C}$ of $\mathrm{Sh}_{H(N)}(G,\mathcal{X})_{\mathbb{C}}$, cf. [Pa, Thm. 2, 2.8 (i)]. The analytic Lie group $\mathrm{Alb}(\mathcal{C})^{\mathrm{an}}$ associated to the albanese variety $\mathrm{Alb}(\mathcal{C})$ is isomorphic to $[\mathrm{Hom}(H^{1,0}(\mathcal{C}(\mathbb{C}),\mathbb{C}),\mathbb{C})]/H_1(\mathcal{C},\mathbb{Z})$ and therefore it is 0. The $\mathbb{Q}$–rank of G^{ad} is 0 and this implies that $\mathrm{Sh}_{H(N)}(G,\mathcal{X})$ is a projective $E(G,\mathcal{X})$-scheme, cf. [BHC, Thm. 12.3 and Cor. 12.4]. From the last two sentences one gets that $\mathcal{C}$ is a connected, projective variety over $\mathbb{C}$ whose albanese variety $\mathrm{Alb}(\mathcal{C})$ is trivial. Thus $\mathcal{C}$ can not be embedded into commutative group varieties over $\mathbb{C}$. Therefore the connected components of the $E(G,\mathcal{X})$-scheme $\mathrm{Sh}_{H(N)}(G,\mathcal{X})$ can not be embedded into torsors of commutative group varieties over $E(G,\mathcal{X})$.

Based on the previous example we get that the class of Néron models introduced below is new (cf. [Va4, Prop. 4.4.1] and [Va11, Thm. 4.3.1]).

Theorem 2. *Suppose that for each prime p that does not divide N and for every prime v of $E(G,\mathcal{X})$ that divides p, we have $e_v \leq p-1$. Suppose that $\mathcal{N}(N)$ is a smooth, projective $O(G,\mathcal{X})[\frac{1}{N}]$-scheme. Then $\mathcal{N}(N)$ is the Néron model of its generic fibre $\mathrm{Sh}_{H(N)}(G,\mathcal{X})$ over $O(G,\mathcal{X})[\frac{1}{N}]$ (and thus it is uniquely determined by $\mathrm{Sh}_{H(N)}(G,\mathcal{X})$ and N).*

Theorem 2 provides a better answer to problem 4.3 (a) than Corollary 1, *provided* in addition we know that $\mathcal{N}(N)$ is a projective $O(G,\mathcal{X})[\frac{1}{N}]$-scheme.

6. Smoothness of integral models

We will use the notations listed at the beginning of Section 5. In this Section we study the smoothness of $\mathcal{N}_v$ and $\mathcal{N}(N)_v$. Let $(G_1,\mathcal{X}_1)$ be a Shimura pair such that the group $G_{1,\mathbb{Q}_p}$ is unramified. Let H_1 be a hyperspecial subgroup of $G_1(\mathbb{Q}_p) = G_{1,\mathbb{Q}_p}(\mathbb{Q}_p)$, cf. Definition 4.1 (f). In 1976 Langlands conjectured the existence of a good integral model of $\mathrm{Sh}_{H_1}(G_1,\mathcal{X}_1)$ over each local ring $O_{(v_1)}$ of $E(G_1,\mathcal{X}_1)$ at a prime v_1 of $E(G_1,\mathcal{X}_1)$ that divides p (see [La, p. 411]); unfortunately, Langlands did not explain what good is supposed to stand for. We emphasize that the assumption that $G_{1,\mathbb{Q}_p}$ is unramified implies that $E(G_1,\mathcal{X}_1)$ is unramified above p (see [Mi3, Cor. 4.7 (a)]); thus the index of ramification e_{v_1} of v_1 is 1.

In 1992 Milne made the following conjecture (slight reformulation made by us, as in [Va1, Conj. 3.2.5]; strictly speaking, both Langlands and Milne stated their conjectures over the completion of $O_{(v_1)}$).

Conjecture 1. *There exists an integral canonical model of $\mathrm{Sh}_{H_1}(G_1,\mathcal{X}_1)$ over $O_{(v_1)}$.*

From the classical works of Zink, Rapoport–Langlands, and Kottwitz one gets (see [Zi1], [LR], and [Ko2]):

Theorem 3. **(a)** *The Milne conjecture holds if* $p \geq 3$ *and* $\mathrm{Sh}(G_1, \mathcal{X}_1)$ *is a Shimura variety of PEL type.*

(b) *The Milne conjecture holds if* $\mathrm{Sh}(G_1, \mathcal{X}_1)$ *is a Shimura variety of PEL type of either A or C type.*

The main results of [Va1] and [Va6] say (see [Va1, 1.4, Thm. 2, and Thm. 6.4.1] and [Va6, Thm. 1.3]):

Theorem 4. *Suppose one of the following two conditions holds:*

(a) $p \geq 5$ *and* $\mathrm{Sh}_{H_1}(G_1, \mathcal{X}_1)$ *is of abelian type;*

(b) p *is arbitrary and* $\mathrm{Sh}_{H_1}(G_1, \mathcal{X}_1)$ *is a unitary Shimura variety.*

Then the Milne conjecture holds. Moreover, for each prime v_1 *of* $E(G_1, \mathcal{X}_1)$ *that divides* p*, the integral canonical model of* $\mathrm{Sh}_{H_1}(G_1, \mathcal{X}_1)$ *over* $O_{(v_1)}$ *is a pro-étale cover of a smooth, quasi-projective* $O_{(v_1)}$*-scheme.*

Remark 2. *See [Va2, Thm. 1.3] and [Va6, Thm. 1.3] for two corrections to the proof of Theorem 4 under the assumption that condition (a) holds. More precisely:*

- *the original argument of Faltings for the proof of Proposition 5 was incorrect and it has been corrected in [Va2] (cf. proof of Proposition 5);*

- *the proof of Theorem 4 for the cases when* $G_{1,\mathbb{C}}^{\mathrm{ad}}$ *has simple factors isomorphic to* $\mathbf{PGL}_{pm}$ *for some* $m \in \mathbb{N}$ *was partially incorrect and it has been corrected by [Va6, Thm. 1.3] (cf. [Va6, Appendix, E.3]).*

6.1. Strategy of the proof of Theorem 4, part a

To explain the four main steps of the proof of the (a) part of Theorem 4, we will sketch the argument why the assumptions that $p > 3$ and that $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$ imply that $\mathcal{N}_p$ is a formally smooth scheme over $\mathbb{Z}_{(p)}$. Let $W(\mathbb{F})$ be the ring of Witt vectors with coefficients in an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Let $B(\mathbb{F}) := W(\mathbb{F})[\frac{1}{p}]$.

Let $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}_p$ and let $z : \mathrm{Spec}(V) \to \mathcal{N}_p$ be a lift of y, where V is a finite, discrete valuation ring extension of $W(\mathbb{F})$. Let e be the index of ramification of V. Let R_e be the p-adic completion of the $W(\mathbb{F})[[x]]$-subalgebra of $B(\mathbb{F})[[x]]$ generated by $\frac{x^{en}}{n!}$ with $n \in \mathbb{N}$. Let Φ_e be the Frobenius endomorphism of R_e which is compatible with the Frobenius automorphism of $W(\mathbb{F})$ and which takes x to x^p. We have a natural $W(\mathbb{F})$-epimorphism $m_\pi : R_e \twoheadrightarrow V$ which maps x to a fixed uniformizer π of V. The kernel J_π of m_π has divided power structures and thus we can speak about the evaluation of F-crystals at the thickening $\mathrm{Spec}(V) \hookrightarrow \mathrm{Spec}(R_e)$ defined naturally by m_π. We now consider the principally quasi-polarized, filtered F-crystal of the pull back (A_V, λ_{A_V}) to $\mathrm{Spec}(V)$ of the universal principally abelian scheme over $\mathcal{M}$ (via $f_p \circ z$). Its evaluation at the thickening $\mathrm{Spec}(V) \hookrightarrow \mathrm{Spec}(R_e)$ is of the form

$$(M_{R_e}, F_V^1, \phi_{M_{R_e}}, \nabla_{M_{R_e}}, \psi_{M_{R_e}}),$$

where M_{R_e} is a free R_e-module of rank $2r$, F^1_V is a direct summand of $H^1_{\mathrm{dR}}(A_V/V) = M_{R_e}/J_\pi M_{R_e}$ of rank r, $\phi_{M_{R_e}}$ is a Φ_e-linear endomorphism of M_{R_e}, $\nabla_{M_{R_e}}$ is an integrable, nilpotent modulo p connection on M_{R_e}, and $\psi_{M_{R_e}}$ is a perfect alternating form on M_{R_e}. The generic fibre of A_V is equipped with a family of Hodge cycles whose de Rham realizations belong to $\mathcal{T}(M_{R_e}[\frac{1}{p}]/J_\pi M_{R_e}[\frac{1}{p}])$ and lift naturally to define a family of tensors $(t_{z,\alpha})_{\alpha\in\mathcal{J}}$ of $\mathcal{T}(M_{R_e}[\frac{1}{p}])$.

The **first main step** is to show that, under some conditions on the closed embedding homomorphism $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathbf{GSp}(L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \psi)$ and under the assumption that $p > 3$, the Zariski closure in $\mathbf{GSp}(M_{R_e}, \psi_{M_{R_e}})$ of the subgroup of $\mathbf{GSp}(M_{R_e}[\frac{1}{p}], \psi_{M_{R_e}})$ that fixes $t_{z,\alpha}$ for all $\alpha \in \mathcal{J}$, is a reductive group scheme $\tilde{G}_{R_e}$ over R_e. See [Va1, Subsect. 5.2] for more details and see [Va1, (5.2.12)] for the fact that the reductive group scheme $\tilde{G}_{R_e}$ is isomorphic to $G_{\mathbb{Z}_{(P)}} \times_{\mathbb{Z}_{(p)}} R_e$.

The **second main step** is to show that we can lift F^1_V to a direct summand $F^1_{R_e}$ of M_{R_e} in such a way that $\psi_{M_{R_e}}(F^1_{R_e} \otimes F^1_{R_e}) = 0$ and that for each element $\alpha \in \mathcal{J}$ the tensor $t_{z,\alpha}$ belongs to the F^0-filtration of $\mathcal{T}(M_{R_e}[\frac{1}{p}])$ defined by $F^1_{R_e}[\frac{1}{p}]$. The essence of this second main step is the classical theory of infinitesimal liftings of cocharacters of smooth group schemes (see [DG, Exp. IX]). Due to the existence of $F^1_{R_e}$, the morphism $z : \mathrm{Spec}(V) \to \mathcal{N}_p$ lifts to a morphism $w : \mathrm{Spec}(R_e) \to \mathcal{N}_p$. The reduction of w modulo the ideal $R_e \cap xB(\mathbb{F})[[x]]$ of R_e is a lift $z_0 : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}_p$ of y. Thus, by replacing z with z_0 we can assume that $V = W(\mathbb{F})$. See [Va1, Subsect. 5.3] for more details.

The **third main step** uses the lift $z_0 : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}_p$ of y and Faltings deformation theory (see [Fa, Sect. 7]) to show that $\mathcal{N}_p$ is formally smooth over $\mathbb{Z}_{(p)}$ at its $\mathbb{F}$-valued point defined by y. See [Va1, Subsect. 5.4] for more details.

The **fourth main step** shows that for $p > 3$ the mentioned conditions on the closed embedding homomorphism $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathbf{GSp}(L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \psi)$ always hold, *provided* we replace $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{S})$ by a suitable other injective map $f_1 : (G_1, \mathcal{X}_1) \hookrightarrow (\mathbf{GSp}(W_1, \psi_1), \mathcal{S}_1)$ with the property that $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}}) = (G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}})$. See [Va1, Subsects. 6.5 and 6.6] for more details.

Remark 3. *In [Va7] it is shown that Theorem 4 holds even if* $p \in \{2, 3\}$ *and* $\mathrm{Sh}(G_1, \mathcal{X}_1)$ *is of abelian type. In [Ki] it is claimed that Theorem 4 holds for* $p \geq 3$. *The work [Ki] does not bring any new conceptual ideas to [Va1], [Va6], and [Va7]. In fact, the note [Ki] is only a variation of [Va1], [Va6], and [Va7]. This variation is made possible due to recent advances in the theory of crystalline representations achieved by Fontaine, Breuil, and Kisin. We emphasize that [Ki] does not work for* $p = 2$ *while [Va7] works as well for* $p = 2$.

6.2. Strategy of the proof of Theorem 4, part b

To explain the three main steps of the proof of the (b) part of Theorem 4, in this Subsection we will assume that $(G_1, \mathcal{X}_1)$ is a simple, adjoint, unitary Shimura pair of isotypic A_n Dynkin type. In [De2, Prop. 2.3.10] it is proved the existence of an injective map $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{S})$ of Shimura pairs such that we have $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}}) = (G_1, \mathcal{X}_1)$.

The **first step** uses a modification of the proof of [De2, Prop. 2.3.10] to show that we can choose f such that $G_{\mathbb{Z}_{(p)}}$ is the subgroup of $\mathbf{GSp}(L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \psi)$ that fixes a semisimple $\mathbb{Z}_{(p)}$–subalgebra $\mathcal{B}_{(p)}$ of $\mathrm{End}(W)$ (see [Va6, Prop. 3.2]). Let $H_{1,p} := G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$; it is a hypersecial subgroup of $G_{1,\mathbb{Q}_p}(\mathbb{Q}_p) = G^{\mathrm{ad}}_{\mathbb{Q}_p}(\mathbb{Q}_p)$.

The **second step** only applies Theorem 3 to conclude that $\mathcal{N}_p$ is a formally smooth $O(G, \mathcal{X})_{(p)}$-scheme.

The **third step** uses the standard moduli interpretation of $\mathcal{N}_p$ to show that the analogue $\mathcal{N}_{1,p}$ of $\mathcal{N}_p$ but for $(G_1, \mathcal{X}_1, H_{1,p})$ instead of for $(G, \mathcal{X}, H_p)$ exists as well (see [Va6, Thm. 4.3 and Cor. 4.4]). If we fix a $\mathbb{Z}_{(p)}$-monomorphism $O(G, \mathcal{X})_{(p)} \hookrightarrow W(\mathbb{F})$, then every connected component $\mathcal{C}_1$ of $\mathcal{N}_{1,W(\mathbb{F})}$ will be isomorphic to the quotient of a connected component $\mathcal{C}$ of $\mathcal{N}_{W(\mathbb{F})}$ by a suitable group action $\mathfrak{T}$ whose generic fibre is free and which involves a torsion group. The key point is to show that the action $\mathfrak{T}$ itself is free (i.e., $\mathcal{C}_1$ is a formally smooth $W(\mathbb{F})$-scheme). If $p > 2$ and p does not divide $n+1$, then the torsion group of the action $\mathfrak{T}$ has no elements of order p and thus the action $\mathfrak{T}$ is free (cf. proof of [Va1, Thm. 6.2.2 b)]). In [Va6] it is checked that the action $\mathfrak{T}$ is always free i.e., it is free even for the harder cases when either $p = 2$ or p divides $n+1$. The proof relies on the moduli interpretation of $\mathcal{N}_p$ which makes this group action quite explicit. The cases $p = 2$ and p divides $n+1$ are the hardest due to the following two reasons.

(i) If $p = 2$ and if A is an abelian variety over $\mathbb{F}$ whose 2-rank a is positive, then the group $(\mathbb{Z}/2\mathbb{Z})^{a^2}$ is naturally a subgroup of the group of automorphisms of the formal deformation space $\mathrm{Def}(A)$ of A in such a way that the filtered Dieudonné module of a lift $\star$ of A to $\mathrm{Spf}(W(\mathbb{F}))$ depends only on the orbit under this action of the $\mathrm{Spf}(W(\mathbb{F}))$-valued point of $\mathrm{Def}(A)$ defined by $\star$.

(ii) For a positive integer m divisible by $p-1$ there exist actions of $Z/p\mathbb{Z}$ on $\mathbb{Z}_p[[x_1, \ldots, x_m]]$ such that the induced actions on $\mathbb{Z}_p[[x_1, \ldots, x_m]][\frac{1}{p}]$ are free.

Theorem 5. *We assume that either* 6 *divides* N *or* $\mathrm{Sh}(G, \mathcal{X})$ *is a unitary Shimura variety. We also assume that the Zariski closure of* G *in* $\mathbf{GL}_{L \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{N}]}$ *is a reductive group scheme over* $\mathbb{Z}[\frac{1}{N}]$. *Then* $\mathcal{N}(N)$ *is a smooth scheme over either* $O(G, \mathcal{X})[\frac{1}{N}]$ *or* $\mathbb{Z}[\frac{1}{N}]$.

Proof: Let p be an arbitrary prime that does not divide N and let v be a prime of $E(G, \mathcal{X})$ that divides p. The group scheme $G_{\mathbb{Z}_{(p)}}$ is reductive. Thus the group $G_{\mathbb{Q}_p}$ is unramified. This implies that $E(G, \mathcal{X})$ is unramified over p and that H_p is a hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Therefore $O(G, \mathcal{X})[\frac{1}{N}]$ is an étale $\mathbb{Z}[\frac{1}{N}]$-algebra. From this and Proposition 3 (a) we get that to prove the Theorem it suffices to show that each scheme $\mathcal{N}_v$ is regular and formally smooth over $O_{(v)}$.

Let $\mathcal{I}_v$ be the integral canonical model of $\mathrm{Sh}_{H_p}(G, \mathcal{X})$ over $O_{(v)}$, cf. Theorem 4. As $\mathcal{I}_v$ is a healthy regular scheme (cf. Theorem 1), from Proposition 4 we get that we have an $O_{(v)}$-morphism $a : \mathcal{I}_v \to \mathcal{N}_v$ whose generic fibre is the identity automorphism of $\mathrm{Sh}_{H_p}(G, \mathcal{X})$. The morphism a is a pro-étale cover of a morphism $a_{H_0} : \mathcal{I}_v/H_0 \to \mathcal{N}_v/H_0$ of normal $O_{(v)}$-schemes of finite type, where H_0 is a small enough compact, open subgroup of $G(\mathbb{A}_f^{(p)})$. From Theorem 4 we get that $\mathcal{I}_v/H_0$ is a quasi-projective $O_{(v)}$-scheme. Thus a_{H_0} is a quasi-projective morphism

between flat $O_{(v)}$-schemes. As each discrete valuation ring of mixed characteristic $(0,p)$ is a healthy regular scheme, the morphism a satisfies the valuative criterion of properness with respect to such discrete valuation rings. From the last two sentences we get that a_{H_0} is in fact a projective morphism.

We consider an open subscheme $\mathcal{V}_v$ of $\mathcal{N}_v$ which contains $\mathrm{Sh}_{H_p}(G,\mathcal{X})$ and for which the morphism $a^{-1}(\mathcal{V}_v) \to \mathcal{V}_v$ is an isomorphism. As $\mathcal{I}_v$ has the extension property (cf. Definition 4.1 (d)), from Theorem 4 we easily get that we can assume that $\mathcal{V}_v$ contains the formally smooth locus of $\mathcal{N}_v$ over $O_{(v)}$. As a_{H_0} is projective, from Proposition 3 (a) we get that we can also assume that we have an inequality $\mathrm{codim}_{\mathcal{N}_v}(\mathcal{N}_v \setminus \mathcal{V}_v) \geq 2$. Obviously we can assume that $\mathcal{V}_v$ is H_0-invariant. Thus the projective morphism $a_{H_0} : \mathcal{I}_v/H_0 \to \mathcal{N}_v/H_0$ is an isomorphism above $\mathcal{V}_v/H_0$.

To check that $\mathcal{N}_v$ is a regular scheme which is formally smooth over $O_{(v)}$ it suffices to show that a_{H_0} is an isomorphism. To check that a_{H_0} is an isomorphism, it suffices to show that $a_{H_0}^{-1}(\mathcal{V}_v/H_0)$ contains all points of $\mathcal{I}_v/H_0$ of codimension 1 (this is so as the projective morphism a_{H_0} is a blowing up of a closed subscheme of $\mathcal{N}_v/H_0$; the proof of this is similar to [Ha, Ch. II, Thm. 7.17]). Let $\mathcal{Y}$ be the set of points of $\mathcal{I}_v/H_0$ which are of codimension 1 and which do not belong to $a_{H_0}^{-1}(\mathcal{V}_v/H_0)\tilde{\to}\mathcal{V}_v/H_0$. We show that the assumption that the set $\mathcal{Y}$ is non-empty leads to a contradiction.

Let $\mathcal{C}$ be the open subscheme of $\mathcal{I}_v/H_0$ that contains: (i) the generic fibre of $\mathcal{I}_v/H_0$ and (ii) the union $\mathcal{E}$ of those connected components of the special fibre of $\mathcal{I}_v/H_0$ whose generic points are in $\mathcal{Y}$. The image $\mathcal{E}_0 := a_{H_0}(\mathcal{E})$ has dimension less than $\mathcal{E}$ and is contained in the non-smooth locus of $\mathcal{N}_v/H_0$. The morphism $\mathcal{C} \to \mathcal{N}_v/H_0$ factors through the dilatation $\mathcal{D}$ of $\mathcal{N}_v/H_0$ centered on the reduced scheme of the non-smooth locus of $\mathcal{N}_v/H_0$, cf. the universal property of dilatations (see Definition 4.1 (g) or [BLR, Ch. 3, 3.2, Prop. 3.1 (b)]). But $\mathcal{D}$ is an affine $\mathcal{N}_v/H_0$-scheme and thus the image of the projective $\mathcal{N}_v/H_0$-scheme $\mathcal{E}$ in $\mathcal{D}$ has the same dimension as $\mathcal{E}_0$. By repeating the process we get that the image of $\mathcal{E}$ in a smoothening $\mathcal{D}_\infty$ of $\mathcal{N}_v/H_0$ obtaining via a sequence of blows up centered on non-smooth loci (see [BLR, Ch. 3, Thm. 3 of 3.1 and Thm. 2 of 3.4]), has dimension $\dim(\mathcal{E}_0)$ and thus it has dimension less than $\mathcal{E}$. But each discrete valuation ring of $\mathcal{D}_\infty$ dominates a local ring of $\mathcal{I}_v/H_0$ (as a_{H_0} is a projective morphism) and therefore it is also a local ring of $\mathcal{I}_v/H_0$. As $\mathcal{D}_\infty$ has at least one discrete valuation ring which is not a local ring of $\mathcal{V}_v/H_0$, we get that this discrete valuation ring is the local ring of some point in $\mathcal{Y}$. Thus the image of $\mathcal{E}$ in $\mathcal{D}_\infty$ has the same dimension as $\mathcal{E}$. Contradiction. $\square$

7. Projectiveness of integral models

The $\mathbb{C}$-scheme $\mathrm{Sh}_{H(N)}(G,\mathcal{X})_{\mathbb{C}}$ is projective if and only if the $\mathbb{Q}$–rank of G^{ad} is 0, cf. [BHC, Thm. 12.3 and Cor. 12.4]. Based on this Morita conjectured in 1975 that (see [Mo]):

Conjecture 2. *Suppose that the $\mathbb{Q}$–rank of G^{ad} is 0. Then for each $N \in \mathbb{N}$ with $N \geq 3$, the $O(G,\mathcal{X})[\frac{1}{N}]$-scheme $\mathcal{N}(N)$ is projective.*

Conjecture 3. *Let A_E be an abelian variety over a number field E. Let H_A be the Mumford–Tate group of some extension A of A_E to $\mathbb{C}$. If the $\mathbb{Q}$–rank of H_A^{ad} is 0, then A_E has potentially good reduction everywhere (i.e., there exists a finite field extension E_1 of E such that A_{E_1} extends to an abelian scheme over the ring of integers of E_1).*

7.1. On the equivalence of Conjectures 2 and 3

In [Mo] it is shown that Conjectures 2 and 3 are equivalent. We recall the argument for this. Suppose that Conjecture 2 holds. To check that Conjecture 3 holds, we can replace E by a finite field extension of it and we can replace A_E by an abelian variety over E which is isogeneous to it. Based on this and [Mu, Ch. IV, §23, Cor. 1], we can assume that A_E has a principal polarization λ_{A_E}. By enlarging E, we can also assume that all Hodge cycles on A are pull backs of Hodge cycles on A_E (cf. [De3, Prop. 2.9 and Thm. 2.11]) and that (A_E, λ_{A_E}) has a level-$l_1 l_2$ symplectic similitude structure. Here l_1 and l_2 are two distinct odd primes. By taking $G = H_A$ and x_A to belong to $\mathcal{X}_A$, we can assume that (A_E, λ_{A_E}) is the pulls back of the universal principally polarized abelian schemes over $\mathcal{N}(l_1)$ and $\mathcal{N}(l_2)$. As $\mathcal{N}(l_1)$ and $\mathcal{N}(l_2)$ are projective schemes over $O(G,\mathcal{X})[\frac{1}{l_1}]$ and $O(G,\mathcal{X})[\frac{1}{l_2}]$ (respectively), we get that (A_E, λ_{A_E}) extends to a principally polarized abelian scheme over the ring of integers of E. Thus Conjecture 2 implies Conjecture 3.

The arguments of the previous paragraph can be reversed to show that Conjecture 3 implies Conjecture 2.

Definition 5. *We say A_E (resp. $(G, \mathcal{X})$) has compact factors, if for each simple factor $\dagger$ of H_A^{ad} (resp. of G^{ad}) there exists a simple factor of $\dagger_{\mathbb{R}}$ which is compact.*

In [Va4, Thm. 1.2 and Cor. 4.3] it is proved that:

Theorem 6. *Suppose that $(G, \mathcal{X})$ (resp. A_E) has compact factors. Then Conjecture 2 (resp. 3) holds.*

7.2. Different approaches

Let $L_A := H_1(A^{\mathrm{an}}, \mathbb{Z})$ and $W_A := L_A \otimes_{\mathbb{Z}} \mathbb{Q}$. We present different approaches to prove Conjectures 2 and 3 developed by Grothendieck, Morita, Paugam, and us.

(a) Suppose that there exists a prime p such that the group $H_{\mathbb{Q}_p}^{\mathrm{ad}}$ is anisotropic (i.e., its $\mathbb{Q}_p$-rank is 0). Then Conjectures 2 and 3 are true (see [Mo] for the potentially good reduction outside of those primes dividing p; see [Pau] for the potentially good reduction even at the primes dividing p).

(b) Let $\mathcal{B}$ be as in Subsubsection 4.2.1 (resp. be the centralizer of H_A in $\mathrm{End}(W_A)$). We assume that the centralizer of $\mathcal{B}$ in $\mathrm{End}(W)$ (resp. in $\mathrm{End}(W_A)$) is a central division algebra over $\mathbb{Q}$. Then Conjecture 2 (resp. 3) holds (see [Mo]).

(c) By replacing E with a finite field extension of it, we can assume that A_E has everywhere semi-abelian reduction. Let $l \in \mathbb{N}$ be a prime different from p. Let $T_l(A_E)$ be the l-adic Tate–module of A_E. As $\mathbb{Z}_l$-modules we can identify

$T_l(A_E) = H_1(A^{\mathrm{an}}, \mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}_l = L_A\otimes_{\mathbb{Z}}\mathbb{Z}_l$. By replacing E with a finite field extension of it, we can assume that for each prime $l \in \mathbb{N}$ the l-adic representation $\rho_l : \mathrm{Gal}(E) \to \mathbf{GL}_{T_l(A_E)}(\mathbb{Q}_l)$ factors through $H_A(\mathbb{Q}_l)$. Let w be a prime of E that divides p. If A_E does not have good reduction at w, then there exists a $\mathbb{Z}_l$-submodule T of $T_l(A_E)$ such that the inertia group of w acts trivially on T and $T_l(A_E)/T$ and non-trivially on $T_l(A_E)$ (see [SGA7, Vol. I, Exp. IX, Thm. 3.5]). This implies that $H_A(\mathbb{Q}_l)$ has unipotent elements of unipotent class 2.

In [Pau] is is shown that if $H_A(\mathbb{Q}_l)$ has no unipotent element of unipotent class 2, then Conjecture 3 holds for A. Using this, Conjectures 2 and 3 are proved in [Pau] in many cases. These cases are particular cases of either Theorem 6 or (a).

(d) We explain the approach used in [Va4] to prove Theorem 6. Let B_E be another abelian variety over E. We say that A_E and B_E are *adjoint-isogeneous*, if the adjoint groups of the Mumford–Tate groups H_A, H_B, and $H_{A\times_{\mathbb{C}}B}$ are isomorphic (more precisely, the standard monomorphism $H_{A\times_{\mathbb{C}}B} \hookrightarrow H_A \times_{\mathbb{Q}} H_B$ induces naturally isomorphisms $H^{\mathrm{ad}}_{A\times_{\mathbb{C}}B} \tilde{\to} H^{\mathrm{ad}}_A$ and $H^{\mathrm{ad}}_{A\times_{\mathbb{C}}B} \tilde{\to} H^{\mathrm{ad}}_B$).

To prove Conjecture 3 for A_E it is the same thing as to prove Conjecture 3 for B_E. Based on this, to prove Conjecture 3, one can replace the monomorphism $H_A \hookrightarrow \mathbf{GL}_{W_A}$ by another one $H_B \hookrightarrow \mathbf{GL}_{W_B}$ which is simpler. Based on this and Subsection 7.1, to prove Conjectures 2 and 3 it suffices to prove Conjecture 2 in the cases when:

(i) the adjoint group G^{ad} is a simple $\mathbb{Q}$–group;

(ii) if F is a totally real number field such that $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}}\, G^{\mathrm{ad},F}$, with $G^{\mathrm{ad},F}$ an absolutely simple adjoint group over F, then F is naturally a $\mathbb{Q}$–subalgebra of the semisimple $\mathbb{Q}$–algebra $\mathcal{B}$ we introduced in Subsubsection 4.2.1;

(iii) the monomorphism $G \hookrightarrow \mathbf{GL}_W$ is simple enough.

Suppose that $(G, \mathcal{X})$ has compact factors. By considering a large field that contains both $\mathbb{R}$ and $\mathbb{Q}_p$, one obtains naturally an identification $\mathrm{Hom}(F, \mathbb{R}) = \mathrm{Hom}(F, \overline{\mathbb{Q}_p})$. Thus we can speak about the p-adic field F_{j_0} which is the factor of

$$F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{j\in J} F_j \tag{23a}$$

that corresponds (via the mentioned identification) to a simple, compact factor of $G^{\mathrm{ad}}_{\mathbb{R}} = \prod_{i\in \mathrm{Hom}(F,\mathbb{R})} G^{\mathrm{ad},F} \times_{F,i} \mathbb{R}$. The existence of such a simple, compact factor is guaranteed by Definition 5.

To prove Theorem 6, it suffices to show that each morphism $c : \mathrm{Spec}(k((x))) \to \mathcal{N}(N)$, with k an algebraically closed field of prime characteristic p that does not divide N, extends to a morphism $\mathrm{Spec}(k[[x]]) \to \mathcal{N}(N)$.

We outline the argument for why the assumption that there exists such a morphism $c : \mathrm{Spec}(k((x))) \to \mathcal{N}(N)$ which does not extend, leads to a contradiction. The morphism c gives birth naturally to an abelian variety E of dimension r over $k((x))$. We can assume that E extends to a semi-abelian scheme $E_{k[[x]]}$ over $k[[x]]$ whose special fibre E_k is not an abelian variety. Let T_k be the maximal torus

of C_k. The field F acts naturally on $X^*(T_k) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $X^*(T_k)$ is the abelian group of characters of T_k. Let k_1 be an algebraic closure of $k((x))$. Let (M, ϕ) be the contravariant Dieudonné module of E_{k_1}. Due to (ii), one has a natural decomposition of $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$-modules

$$(23b) \qquad (M[\frac{1}{p}], \phi) = \oplus_{j \in J}(M_j, \phi).$$

For each $m \in \mathbb{N}$, the composite monomorphism $T_k[p^m] \hookrightarrow E_k[p^m] \hookrightarrow E_k$ lifts uniquely to a homomorphism $(T_k[p^m])_{k[[x]]} \to E_{k[[x]]}$ (see [DG, Exp. IX, Thms. 3.6 and 7.1]) which due to Nakayama's lemma is a closed embedding. This implies that we have a monomorphism $(T_k[p^m])_{k((x))} \hookrightarrow E[p^m]$. Taking $m \to \infty$, at the level of Dieudonné modules over k_1 we get an epimorphism

$$(23c) \qquad \theta : (M[\frac{1}{p}], \phi) \twoheadrightarrow (X^*(T_k) \otimes_{\mathbb{Z}} B(k_1), 1_{X^*(T_k)} \otimes p\sigma_{k_1})$$

which (due to the uniqueness part of this paragraph) is compatible with the natural F-actions. Here σ_{k_1} is the Frobenius automorphism of the field of fractions $B(k_1)$ of the ring $W(k_1)$ of Witt vectors with coefficients in k_1.

From (23b) and (23c) we get that each (M_j, ϕ) has Newton polygon slope 1. But based on (iii) one can assume that the F-isocrystal (M_{j_0}, ϕ) has no integral Newton polygon slope. Contradiction.

8. Stratifications

We will use the notations listed in the beginning of Section 5. Let $\mathcal{N}(N)_v^{\mathrm{s}}$ be the smooth locus of $\mathcal{N}(N)_v$ over $O_{(v)}$; its generic fibre is $\mathrm{Sh}_{H(N)}(G, \mathcal{X})$ (cf. Subsection 4.2). In this Section we will study different stratifications of the special fibre $\mathcal{L}(N)_v^{\mathrm{s}}$ of $\mathcal{N}(N)_v^{\mathrm{s}}$. We begin with few extra notations.

Let ψ^* be the perfect alternating form on L^* induced naturally by ψ. Let $\mathcal{H}_{\mathbb{Z}_{(p)}}$ be the flat, closed subgroup scheme of $G_{\mathbb{Z}_{(p)}}$ which fixes ψ^*; its generic fibre is a connected group $\mathcal{H}_{\mathbb{Q}}$. Let $(s_\alpha)_{\alpha \in \mathcal{J}} \subseteq \mathcal{T}(W^*)$ be a family of tensors as in Subsection 3.4.2. We denote also by $(\mathcal{V}, \Lambda_{\mathcal{V}})$ the pull back to $\mathcal{N}(N)$ of the universal principally polarized abelian scheme over $\mathcal{A}_{r,1,N}$ (to be compared with Subsubsection 3.4.2). By replacing N with an integral power of itself, we can speak about a family $(v_\alpha^{\mathcal{V}})_{\alpha \in \mathcal{J}}$ of Hodge cycles on $\mathcal{V}_{\mathbb{Q}}$ obtained as in Subsubsection 3.4.2. Such a replacement is irrelevant for this Section as we are interested in points of $\mathcal{N}(N)$ with values in k, $W(k)$, or $B(k)$. Here k is an algebraically closed field of characteristic p, $W(k)$ is the ring of Witt vectors with coefficients in k, and $B(k) = W(k)[\frac{1}{p}]$ is the field of fractions of $W(k)$. Let σ_k be the Frobenius automorphism of k, $W(k)$, or $B(k)$.

All the results of Section 5 involve finite primes unramified over p. Due to this in this Section we will assume that:

(*) the prime v of $E(G, \mathcal{X})$ is unramified over p and the $k(v)$-scheme $\mathcal{L}(N)_v^{\mathrm{s}}$ is non-empty.

See [Va7, Lem. 4.1] for a general criterion on when (*) holds.

8.1. F-crystals with tensors

Let $y : \mathrm{Spec}(k) \to \mathcal{L}(N)^{\mathrm{s}}_v$. Let $z : \mathrm{Spec}(W(k)) \to \mathcal{N}(N)^{\mathrm{s}}_v$ be a lift of y, cf. (*). Let $(A, \lambda_A) := z^*((\mathcal{V}, \Lambda_{\mathcal{V}})_{\mathcal{N}(N)^{\mathrm{s}}_v})$. Let

$$(M, \phi, \psi_M)$$

be the principally quasi-polarized Dieudonné module of $(A, \lambda_A)_k$. Thus ψ_M is a perfect alternating form on M such that we have $\psi(\phi(a) \otimes \phi(b)) = p\sigma_k(\psi(a \otimes b))$ for all $a, b \in M$. The σ_k-linear automorphism $\phi : M[\frac{1}{p}] \tilde{\to} M[\frac{1}{p}]$ extends naturally to a σ_k-linear automorphism $\phi : \mathcal{T}(M[\frac{1}{p}]) \tilde{\to} \mathcal{T}(M[\frac{1}{p}])$.

The abelian variety $A_{B(k)}$ is endowed naturally with a family $(v_\alpha)_{\alpha \in \mathcal{J}}$ of Hodge cycles (it is obtained from the family $(v^{\mathcal{V}}_\alpha)_{\alpha \in \mathcal{J}}$ of Hodge cycles on $\mathcal{V}_{\mathbb{Q}}$ via a natural pull back process). Let $t_\alpha \in \mathcal{T}(M[\frac{1}{p}])$ be the de Rham component of v_α.

Let F^1 be the Hodge filtration of M defined by the lift A of A_k. We have $\phi(\frac{1}{p}F^1 + M) = M$. Let $\mu_z : \mathbb{G}_m \to \mathbf{GL}_M$ be the inverse of the canonical split cocharacter of (M, F^1, ϕ) defined in [We, p. 512]. It gives birth to a direct sum decomposition $M = F^1 \oplus F^0$ such that $\mathbb{G}_m$ acts via μ_z trivially on F^0 and via the inverse of the identical character of $\mathbb{G}_m$ on F^1.

It is known that the element t_α of $\mathcal{T}(M[\frac{1}{p}])$ is a de Rham and thus also crystalline cycle. If the abelian variety $A_{B(k)}$ is definable over a number subfield of $B(k)$, then this result was known since long time (for instance, see [Bl, Thm. (0.3)]). The general case follows from loc. cit. and [Va1, Principle B of 5.2.16] (in the part of [Va1, Subsect. 5.2] preceding the Principle B an odd prime is used; however the proof of loc. cit. applies to all primes). The fact that t_α is a crystalline cycle means that:

(i) the tensor t_α belongs to the F^0-filtration of $\mathcal{T}(M[\frac{1}{p}])$ defined by $F^1[\frac{1}{p}]$ and it is fixed by ϕ.

Let $\mathcal{G}_{B(k)}$ be the subgroup of $\mathbf{GSp}(M[\frac{1}{p}], \psi_M)$ that fixes t_α for all $\alpha \in \mathcal{J}$. Let $\mathcal{G}$ be the Zariski closure of $\mathcal{G}_{B(k)}$ in $\mathbf{GSp}(M, \psi_M)$ (or $\mathbf{GL}_M$); it is an affine, flat group scheme over $W(k)$. We refer to the quadruple

$$\mathcal{C}_y := (M, \phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M)$$

as the *principally quasi-polarized F-crystal with tensors* attached to $y \in \mathcal{N}(N)^{\mathrm{s}}_v$. It is easy to see that this terminology makes sense (i.e., t_α depends only on $y : \mathrm{Spec}(k) \to \mathcal{L}(N)^{\mathrm{s}}_v$ and not on the choice of the lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}(N)^{\mathrm{s}}_v$ of y). We note down that $\mathcal{G}$ is uniquely determined by $\mathcal{C}_y$. We refer to the quadruple

$$\mathcal{R}_y := (M[\frac{1}{p}], \phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M)$$

as the *principally quasi-polarized F-isocrystal with tensors* attached to $y \in \mathcal{N}(N)^{\mathrm{s}}_v$. From (i) and the functorial aspects of [Wi, p. 513] we get that each tensor t_α is fixed by μ_z. This implies that:

(ii) the cocharacter $\mu_z : \mathbb{G}_m \to \mathbf{GL}_M$ factors through $\mathcal{G}$ and we denote also by $\mu_z : \mathbb{G}_m \to \mathcal{G}$ this factorization.

If $y_i : \mathrm{Spec}(k) \to \mathcal{L}(N)_v^{\mathrm{s}}$ is a point indexed by the elements i of some set, then we will use the index i to write down $\mathcal{C}_{y_i} = (M_i, \phi_i, (t_{i,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_i})$ as well as $\mathcal{R}_{y_i} = (M_i[\frac{1}{p}], \phi_i, (t_{i,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_i})$.

If $y_i : \mathrm{Spec}(k) \to \mathcal{L}(N)_v$ does not factor through $\mathcal{L}(N)_v^{\mathrm{s}}$, then we define $\mathcal{C}_{y_i} := (M_i, \phi_i, \psi_{M_i})$ to be the principally quasi-polarized Dieudonné module of $y_i^*((\mathcal{V}, \Lambda_{\mathcal{V}})_{\mathcal{L}(N)_v^{\mathrm{s}}})$. Similarly we define $\mathcal{R}_{y_i} := (M_i[\frac{1}{p}], \phi_i, \psi_{M_i})$.[1]

Before studying different stratifications of $\mathcal{L}(N)_v^{\mathrm{s}}$ defined naturally by basic properties of the $\mathcal{C}_y$'s, we will first present basic definitions on stratifications of reduced schemes over fields.

8.2. Types of stratifications

Let K be a field. By a *stratification* $\mathfrak{S}$ of a reduced $\mathrm{Spec}(K)$-scheme X (*in potentially an infinite number of strata*), we mean that:

(i) for each field l that is either K or an algebraically closed field which contains K and that has countable transcendental degree over K, a set $\mathfrak{S}_l$ of disjoint reduced, locally closed subschemes of X_l is given such that each point of X_l with values in an algebraic closure of l factors through some element of $\mathfrak{S}_l$;

(ii) if $i_{12} : l_1 \hookrightarrow l_2$ is an embedding between two fields as in (a), then the reduced scheme of the pull back to l_2 of every member of $\mathfrak{S}_{l_1}$, is an element of $\mathfrak{S}_{l_2}$; thus we have a natural pull back injective map $\mathfrak{S}(i_{12}) : \mathfrak{S}_{l_1} \hookrightarrow \mathfrak{S}_{l_2}$.

Each element $\mathfrak{s}$ of some set $\mathfrak{S}_l$ is referred as a *stratum* of $\mathfrak{S}$; we denote by $\bar{\mathfrak{s}}$ the Zariski closure of $\mathfrak{s}$ in X_l. If all maps $\mathfrak{S}(i_{12})$'s are bijections, then we identify $\mathfrak{S}$ with $\mathfrak{S}_K$ and we say $\mathfrak{S}$ is *of finite type*.

Definition 6. *We say that the stratification $\mathfrak{S}$ has (or satisfies):*

(a) *the strong purity property if for each field l as in (i) above and for every stratum $\mathfrak{s}$ of $\mathfrak{S}_l$, locally in the Zariski topology of $\bar{\mathfrak{s}}$ we have $\mathfrak{s} = \bar{\mathfrak{s}}_a$, where a is some global function of $\bar{\mathfrak{s}}$ and where $\bar{\mathfrak{s}}_a$ is the largest open subscheme of $\bar{\mathfrak{s}}$ over which a is an invertible function;*

(b) *the purity property if for each field l as in (i) above, every element of $\mathfrak{S}_l$ is an affine X_l-scheme (thus $\mathfrak{S}$ has the purity property if and only if each stratum of it is an affine X-scheme);*

(c) *the weak purity property if for each field l as in (i) above and for every stratum $\mathfrak{s}$ of $\mathfrak{S}_l$, the scheme $\bar{\mathfrak{s}}$ is noetherian and the complement of $\mathfrak{s}$ in $\bar{\mathfrak{s}}$ is either empty or has pure codimension 1 in $\bar{\mathfrak{s}}$.*

[1] For each lift of y_i to a point of $\mathcal{N}(N)_v$ with values in a finite discrete valuation ring extension of $W(k)$, one defines naturally a family of tensors $(t_{i,\alpha})_{\alpha\in\mathcal{J}}$ of $\mathcal{T}(M_i[\frac{1}{p}])$. We do not know if this family of tensors: (i) does not depend on the choice of the lift and (ii) can be used in Subsections 8.4 and 8.5 in the same way as the families of tensors attached to k-valued points of $\mathcal{L}(N)_v^{\mathrm{s}}$.

As the terminology suggests, the strong purity property implies the purity property and the purity property implies the weak purity property. The converses of these two statements do not hold. For instance, there exist affine, integral, noetherian schemes Y which have open subschemes whose complements in Y have pure codimension 1 in Y but are not affine (see [Va3, Rm. 6.3 (a)]).

8.2.1. Example

Suppose that $K = k$, that X is an integral k-scheme, and that there exists a Barsotti–Tate group D of level 1 over X which generically is ordinary. Let $\mathfrak{O}$ be the stratification of X of finite type which has two strata: the ordinary locus $\mathfrak{s}_0$ of D and the non-ordinary locus $\mathfrak{s}_{\mathrm{n}}$ of D. We have $\bar{\mathfrak{s}}_{\mathrm{o}} = X$ and $\bar{\mathfrak{s}}_{\mathrm{n}} = \mathfrak{s}_{\mathrm{n}}$. Moreover locally in the Zariski topology of X we have an identity $\mathfrak{s}_{\mathrm{o}} = X_a$, where a is the global function on X which is the determinant of the Hasse–Witt map of D. Thus the stratification $\mathfrak{O}$ has the strong purity property.

8.3. Newton polygon stratification

Let $\mathfrak{N}$ be the stratification of $\mathcal{L}(N)_v$ of finite type with the property that two geometric points $y_1, y_2 : \mathrm{Spec}(k) \to \mathcal{L}(N)_v$ factor through the same stratum if and only if the Newton polygons of (M_1, ϕ_1) and (M_2, ϕ_2) coincide. In [dJO2] it is shown that $\mathfrak{N}$ has the weak purity property (see [Zi2] for a more recent and nice proof of this).

Theorem 7. *The stratification $\mathfrak{N}$ of $\mathcal{L}(N)_v$ has the purity property.*

Proof: The stratification $\mathfrak{N}$ is the Newton polygon stratification of $\mathcal{L}(N)_v$ defined by the F-crystal over $\mathcal{L}(N)_v$ associated to the p-divisible group of $\mathcal{V}_{\mathcal{L}(N)_v}$. Thus the Theorem is a particular case of [Va3, Main Thm. B]. □

8.4. Rational stratification

Let $\mathfrak{R}$ be the stratification of $\mathcal{L}(N)_v^{\mathrm{s}}$ with the property that two geometric points $y_1, y_2 : \mathrm{Spec}(k) \to \mathcal{L}(N)_v^{\mathrm{s}}$ factor through the same stratum if and only if there exists an isomorphism $\mathcal{R}_{y_1} \tilde{\to} \mathcal{R}_{y_2}$ to be called a *rational isomorphism.*

Theorem 8. *The following three properties hold:*

(a) *Each stratum of $\mathfrak{R}$ is an open closed subscheme of a stratum of the restriction $\mathfrak{N}^{\mathrm{s}}$ of $\mathfrak{N}$ to $\mathcal{L}(N)_v^{\mathrm{s}}$.*

(b) *The stratification $\mathfrak{R}$ of $\mathcal{L}(N)_v^{\mathrm{s}}$ is of finite type.*

(c) *The stratification $\mathfrak{R}$ of $\mathcal{L}(N)_v^{\mathrm{s}}$ has the purity property.*

Proof: We use left lower indices to denote pulls back of F-crystals. Let l be either $k(v)$ or an algebraically closed field that contains $k(v)$ and that has countable transcendental degree over $k(v)$. Let S_0 be a stratum of $\mathfrak{N}^{\mathrm{s}}$ contained in $\mathcal{L}(N)_{v,l}^{\mathrm{s}}$. Let S_1 be an irreducible component of S_0. To prove the part (a) it suffices to show that for each two geometric points y_1 and y_2 of S_1 with values in the same

algebraically closed field k, there exists a rational isomorphism $\mathcal{R}_{y_1}\tilde{\to}\mathcal{R}_{y_2}$. We can assume that k is an algebraic closure of $\bar{l}((x))$ and that y_1 and y_2 factor through the generic point and the special point (respectively) of a morphism $m : \mathrm{Spec}(\bar{l}[[x]]) \to \mathcal{L}(N)^{\mathrm{s}}_{v,l}$ of l-schemes. Here x is an independent variable. We denote also by y_1 and y_2, the k-valued points of $\mathrm{Spec}(\bar{l}[[x]])$ or of its perfection $\mathrm{Spec}(\bar{l}[[x]]^{\mathrm{perf}})$ defined naturally by the factorizations of y_1 and y_2 through m.

Let Φ be the Frobenius lift of $W(\bar{l})[[x]]$ that is compatible with $\sigma_{\bar{l}}$ and that takes x to x^p. Let $\mathfrak{V} = (V, \phi_V, \psi_V, \nabla_V)$ be the principally quasi-polarized F-crystal over $\bar{l}[[x]]$ of $m^*((\mathcal{V}, \Lambda_{\mathcal{V}})_{\mathcal{L}(N)^{\mathrm{s}}_{v,l}})$. Thus V is a free $W(\bar{l})[[x]]$-module of rank $2r$ equipped with a perfect alternating form ψ_V, $\phi_V : V \to V$ is a Φ-linear endomorphism, and $\nabla_V : V \to V dx$ is a connection. Let $t^V_\alpha \in \mathcal{T}(V[\frac{1}{p}])$ be the de Rham realization of the Hodge cycle $n^*_{B(\bar{l})}(v^{\mathcal{V}}_\alpha)$ on $n^*_{B(k)}((\mathcal{V})_{\mathcal{N}(N)^{\mathrm{s}}_{W(l)}})$, where $n : \mathrm{Spec}(W(\bar{l})[[x]]) \to \mathcal{N}(N)^{\mathrm{s}}_{W(l)}$ is a lift of m.

Fontaine's comparison theory (see [Fo]) assures us that there exists an isomorphism $(M_1[\frac{1}{p}], (t_{1,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_1})\tilde{\to}(W^* \otimes_{\mathbb{Q}} B(k), (s_\alpha)_{\alpha\in\mathcal{J}}, \psi^*)$.

Based on this and [Ko1] we get that $\mathcal{R}_{y_1}$ is isomorphic to the pull back to k of the principally quasi-polarized F-isocrystal $\mathcal{R}_1$ with tensors defined naturally by a principally quasi-polarized F-crystal $\mathcal{C}_1$ with tensors over an algebraic closure $\overline{k(v)}$ of $k(v)$. Strictly speaking [Ko1] uses a language of σ_k-conjugacy classes of sets of the form $G(B(k))$ or $\mathcal{H}_{\mathbb{Q}}(B(k))$ and not a language which involves polarizations and tensors (and thus which involves σ_k-conjugacy classes of sets of the form $\mathcal{H}_{\mathbb{Q}}(B(k))s_0$, where $s_0 \in G(B(k))$ is an element whose image in $(G/\mathcal{H}_{\mathbb{Q}})(B(k)) = \mathbb{G}_m(B(k))$ belongs to $(G/\mathcal{H}_{\mathbb{Q}})(\mathbb{Q}_p) = \mathbb{G}_m(\mathbb{Q}_p)$); but the arguments of [Ko1] apply entirely in the present principally quasi-polarized context which involves sets of the form $\mathcal{H}_{\mathbb{Q}}(B(k))s_0 \subseteq G(B(k))$. Here s_0 is $\mu_0[\frac{1}{p}]$, where $\mu_0 : \mathbb{G}_m \to G_{B(k)}$ is an arbitrary cocharacter whose extension to $\mathbb{C}$ via an $O_{(v)}$-monomorphism $B(k) \hookrightarrow \mathbb{C}$ is $G(\mathbb{C})$-conjugate to the cocharacters $\mu_x : \mathbb{G}_m \to G_{\mathbb{C}}$ with $x \in \mathcal{X}$.

Let $\mathcal{C}^-_1$ be $\mathcal{C}_1$ but viewed only as an F-crystal over $\overline{k(v)}$. Let $M_{1,1}$ be the $W(k)$-lattice of $M_1[\frac{1}{p}]$ that corresponds naturally to $\mathcal{C}^-_{1,k}$ via an isomorphism $i_{1,1} : \mathcal{R}_{1,k}\tilde{\to}\mathcal{R}_{y_1}$.

From [Ka, Thm. 2.7.4] we get the existence of an isogeny $i_0 : \mathfrak{V}_0 \to \mathfrak{V}$, where $\mathfrak{V}_0$ is an F-crystal over $\bar{l}[[x]]$ whose extension to the $\bar{l}[[x]]$-subalgebra $\bar{l}[[x]]^{\mathrm{perf}}$ of k is constant (i.e., it is the pull back of an F-crystal over $\bar{l}$). Let $i_{0,k} : M_0 \to M_1$ be the $W(k)$-linear monomorphism that defines $y_1^*(i_0)$. We can assume that $i_{0,k}(M_0)$ is contained in $M_{1,1}$. The inclusion $i_{0,k}(M_0) \subseteq M_{1,1}$ gives birth to a morphism $j_0 : \mathfrak{V}_{0,k} \to \mathcal{C}^-_{1,k}$ of F-crystals over k. It is the extension to k of a morphism $j_0^{\mathrm{perf}} : \mathfrak{V}_{0,\bar{l}[[x]]^{\mathrm{perf}}} \to \mathcal{C}^-_{1,\bar{l}[[x]]^{\mathrm{perf}}}$, cf. [RR, Lem. 3.9] and the fact that $\mathfrak{V}_{0,\bar{l}[[x]]^{\mathrm{perf}}}$ and $\mathcal{C}^-_{1,\bar{l}[[x]]^{\mathrm{perf}}}$ are constant F-crystals over $k[[x]]^{\mathrm{perf}}$. Let $j_1^{\mathrm{perf}} : \mathcal{C}^-_{1,\bar{l}[[x]]^{\mathrm{perf}}} \to \mathfrak{V}_{0,\bar{l}[[x]]^{\mathrm{perf}}}$ be a morphism of F-crystal such that $j_0^{\mathrm{perf}} \circ j_1^{\mathrm{perf}} = p^q 1_{\mathcal{C}^-_{1,\bar{l}[[x]]^{\mathrm{perf}}}}$ for some $q \in \mathbb{N}$. By composing j_1^{perf} with $i_{0,k[[x]]^{\mathrm{perf}}}$ we get an isogeny $i_1 : \mathcal{C}^-_{1,\bar{l}[[x]]^{\mathrm{perf}}} \to \mathfrak{V}_{\bar{l}[[x]]^{\mathrm{perf}}}$ whose extension to k is defined by the inclusion $p^q M_{1,1} \subseteq M_1$. The isomorphism of F-isocrystals over $\mathrm{Spec}(k[[x]]^{\mathrm{perf}})$ defined by p^{-q} times i_1 takes $t_{1,\alpha}$ to t^V_α for all $\alpha \in \mathcal{J}$, as this is so generically. Thus $y_2^*(i_1)$ is an isogeny which when viewed as an isomorphism of F-isocrystals is p^q times an isomorphism $i_{1,2} : \mathcal{R}_{1,k}\tilde{\to}\mathcal{R}_{y_2}$. Thus there exists a rational isomorphism $i_{1,2} \circ i^{-1}_{1,1} : \mathcal{R}_{y_1}\tilde{\to}\mathcal{R}_{y_2}$. Thus (a) holds.

Part (b) follows from (a) and the fact that $\mathfrak{N}^{\mathrm{s}}$ is a stratification of finite type. Part (c) follows from (a) and Theorem 7. $\square$

Remark 4. *The proof of Theorem 8 (a) and (b) is in essence only a concrete variant of a slight refinement of [RR, Thm. 3.8]. The only new thing it brings to loc. cit., is that it weakens the hypotheses of loc. cit. (i.e., it considers the "Newton point" of only one faithful representation which is $G_{\mathbb{Q}_p} \hookrightarrow \boldsymbol{GL}_{W^*\otimes_{\mathbb{Q}}\mathbb{Q}_p}$).*

8.5. A quasi Shimura p-variety of Hodge type

Let $\mathcal{H} := \mathcal{H}_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} W(k)$, where $\mathcal{H}_{\mathbb{Z}_{(p)}}$ is as in the beginning of this Section. The group $\mathcal{H}_{B(k)}$ is a connected group and we have a short exact sequence

$$1 \to \mathcal{H} \to G_{W(k)} \to \mathbb{G}_m \to 1. \tag{24}$$

We fix an $O_{(v)}$-embedding $W(k) \hookrightarrow \mathbb{C}$. Let ν be the set of cocharacters of $G_{W(k)}$ whose extension to $\mathbb{C}$ are $G(\mathbb{C})$-conjugate to any one of the cocharacters $\mu_x : \mathbb{G}_m \to G_{\mathbb{C}}$ with $x \in \mathcal{X}$. Let $\mu_z : \mathbb{G}_m \to \mathcal{G}$ be the cocharacter introduced in the property (ii) of Subsection 8.1.

Until the end we will also assume that the following three properties hold:

(**) the group scheme $G_{\mathbb{Z}_{(p)}}$ is smooth over $\mathbb{Z}_{(p)}$;

(***) for each algebraically closed field k of countable transcendental degree over $k(v)$ and for every point $y : \mathrm{Spec}(k) \to \mathcal{L}(N)^{\mathrm{s}}_v$, there exists an isomorphism

$$\rho_y : (M_0, (s_\alpha)_{\alpha\in\mathcal{J}}, \psi^*) \tilde{\to} (M, (t_\alpha)_{\alpha\in\mathcal{J}}, \psi_M);$$

(****) the set ν_0 of cocharacters of $G_{W(k)}$ formed by all cocharacters of the form $\rho_y^{-1}\mu_z\rho_y : \mathbb{G}_m \to G_{W(k)}$, with z running through all $W(k)$-valued points of $\mathcal{N}(N)^{\mathrm{s}}$ and with ρ_y running through all isomorphisms as in (**), is a $\mathcal{H}(W(k))$-conjugacy class of cocharacters of $G_{W(k)}$.

We fix an element $\mu_0 : \mathbb{G}_m \to G_{W(k)}$ of ν_0. Let

$$\mathcal{E}_0 := (M_0, \phi_0, (s_\alpha)_{\alpha\in\mathcal{J}}, \psi^*) := (L^* \otimes_{\mathbb{Z}} W(k), \mu_0(\frac{1}{p})(1 \otimes \sigma), \mathcal{H}, (s_\alpha)_{\alpha\in\mathcal{J}}, \psi^*)$$

Let $\vartheta_0 : M_0 \to M_0$ be the Verschiebung map of ϕ_0. We have $\vartheta_0\phi_0 = \phi_0\vartheta_0 = p1_{M_0}$.

Let $[x_z, g_z] \in \mathrm{Sh}_{H(N)}(G, \mathcal{X})(\mathbb{C})$ be the complex point defined by the composite of the morphism $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(W(k))$ with $z : \mathrm{Spec}(W(k)) \to \mathcal{N}(N)^{\mathrm{s}}_v$. Under the fixed $O_{(v)}$-embedding $W(k) \hookrightarrow \mathbb{C}$, we can identify:

– $M \otimes_{W(k)} \mathbb{C} = H^1_{\mathrm{dR}}(A^{\mathrm{an}}/\mathbb{C}) = H^1(A^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = W^* \otimes_{\mathbb{Q}} \mathbb{C}$ (cf. property (d) of Subsubsection 3.4.1 for the last identification);

– $F^1 \otimes_{W(k)} \mathbb{C}$ with the Hodge filtration of $H^1_{\mathrm{dR}}(A^{\mathrm{an}}/\mathbb{C}) = W^* \otimes_{\mathbb{Q}} \mathbb{C}$ defined by the point $x_z \in \mathcal{X}$;

– $t_\alpha = s_\alpha$ for all $\alpha \in \mathcal{J}$ and thus $\mathcal{G}_{\mathbb{C}} = G_{\mathbb{C}}$ (see [De3]).

Based on this we easily get that $\mu_{z,\mathbb{C}} : \mathbb{G}_m \to \mathcal{G}_{\mathbb{C}} = G_{\mathbb{C}}$ is $\mathbb{G}(\mathbb{C})$-conjugate to $\mu_{x_z} : \mathbb{G}_m \to G_{\mathbb{C}}$. From this we get that the cocharacter $\rho_y^{-1}\mu_z\rho_y : \mathbb{G}_m \to G_{W(k)}$ belongs to ν. Thus we have:

$$\nu_0 \subseteq \nu.$$

By composing ρ_y with an automorphism of $(L^* \otimes_{\mathbb{Z}} W(k), (s_\alpha)_{\alpha\in\mathcal{J}})$ defined by an element of $\mathcal{H}(W(k))$, we can assume that in fact we have $\rho_y^{-1}\mu_z\rho_y = \mu_0$ (cf. (****)). This implies that ρ_y gives birth to an isomorphism of the form

$$\rho_y : (M_0, g_y\phi_0, \mathcal{H}, (s_\alpha)_{\alpha\in\mathcal{J}}, \psi^*) \tilde{\to} \mathcal{C}_y$$

for some element $g_y \in G_{W(k)}(W(k))$. For $g \in \mathcal{H}(W(k))$, let

$$\mathcal{E}_g := (M_0, g\phi_0, (s_\alpha)_{\alpha\in\mathcal{J}}, \psi^*).$$

Therefore $\mathcal{E}_0 = \mathcal{E}_{1_{M_0}}$ and moreover

$$\mathcal{C}_y \text{ is isomorphic with } \mathcal{E}_{g_y}.$$

Let

$$\mathcal{F}_0 := \{\mathcal{E}_g | g \in \mathcal{H}(W(k))\};$$

it is a *family* of principally quasi-polarized F-crystals with tensors. We emphasize that, due to (***) and (****), the isomorphism class of the family $\mathcal{F}_0$ depends only on $\mathcal{L}(N)_v^{\mathrm{s}}$ and not on the choice of the element $\mu_0 : \mathbb{G}_m \to G_{W(k)}$ of ν_0.

Definition 7. *Let $m \in \mathbb{N}$. By the D-truncation of level m (or mod p^m) of $\mathcal{E}_g$ we mean the reduction $\mathcal{E}_g[p^m]$ of $(M_0, g\phi_0, \vartheta_0 g^{-1}, \mathcal{H}, \psi^*)$ modulo p^m (here it is more convenient to use $\mathcal{H}$ instead of $(s_\alpha)_{\alpha\in\mathcal{J}}$). For $g_1, g_2 \in \mathcal{H}(W(k))$, by an inner isomorphism between $\mathcal{E}_{g_1}[p^m]$ and $\mathcal{E}_{g_1}[p^m]$ we mean an isomorphism $\mathcal{E}_{g_1}[p^m] \tilde{\to} \mathcal{E}_{g_1}[p^m]$ defined by an element of $\mathcal{H}(W_m(k))$.*

Remark 5. (a) *Statement (***) is a more general form of a conjecture of Milne (made in 1995). In [Va8] it is shown that (***) holds if either $p > 2$ or $p = 2$ and $G_{\mathbb{Z}_{(p)}}$ is a torus. The particular case of this result when moreover $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$, is also claimed in [Ki].*

(b) *If the statement (****) does not hold, then one has to work out what follows with a fixed connected component of $\mathcal{L}(N)_v^{\mathrm{s}}$ instead of with $\mathcal{L}(N)_v^{\mathrm{s}}$.*

(c) *In many cases one can choose the cocharacter $\mu_0 : \mathbb{G}_m \to G_{W(k)}$ in such a way that the quadruple $\mathcal{E}_0$ is a canonical object of the family $\mathcal{F}_0$. For instance, if $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$, then we have $\nu_0 = \nu$ and one can choose μ_0 as follows. Let $B_{\mathbb{Z}_p}$ be a Borel subgroup scheme of $G_{\mathbb{Z}_p} := G_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$. Let $T_{\mathbb{Z}_p}$ be a maximal torus of $B_{\mathbb{Z}_p}$. Let $G_{W(k(v))} := G_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} W(k(v))$. Let $\mu_{0,W(k(v)} : \mathbb{G}_m \to G_{W(k(v))}$ be the unique cocharacter whose extension $\mu_0 : \mathbb{G}_m \to G_{W(k)}$ to $W(k)$ belongs to the set ν, which factors through $T_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} W(k(v))$, and*

through which $\mathbb{G}_m$ acts on $\mathrm{Lie}(B_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k(v))$ via the trivial and the identical characters of $\mathbb{G}_m$ (cf. [Mi3, Cor. 4.7 (b)]). As pairs of the form $(B_{\mathbb{Z}_p}, T_{\mathbb{Z}_p})$ are $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$-conjugate, the isomorphism class of $\mathcal{E}_0$ constructed via such a cocharacter μ_0 does not depend on the choice of $(B_{\mathbb{Z}_p}, T_{\mathbb{Z}_p})$. Thus $\mathcal{E}_0$ is a canonical object of the family $\mathcal{F}_0$.

Theorem 9. *Under the assumptions (*) to (****) of this Section, the $\mathcal{A}_{r,1,N,k(v)}$-scheme $\mathcal{L}(N)^{\mathrm{s}}_v$ is a quasi Shimura p-variety of Hodge type relative to $\mathcal{F}_0$ in the sense of [Va5, Def. 4.2.1].*

Proof: As [Va5, Def. 4.2.1] is a very long definition, the essence of its parts will be pointed out at the right time in this proof. We emphasize that due to (**) and (24), the group scheme $\mathcal{H}$ is smooth over $W(k)$ and therefore the statement of the Theorem makes sense.

Let

$$M_0 = F_0^1 \oplus F_0^0 \tag{25a}$$

be the direct sum decomposition such that $\mathbb{G}_m$ acts through $\mu_0 : \mathbb{G}_m \to G_{W(k)}$ trivially on F_0^0 and via the inverse of the identical character of $\mathbb{G}_m$ on F_0^1. To (25a) corresponds a direct sum decomposition

$$\mathrm{End}(M_0) = \mathrm{Hom}(F_0^0, F_0^1) \oplus \mathrm{End}(F_0^0) \oplus \mathrm{End}(F_0^1) \oplus \mathrm{Hom}(F_0^1, F_0^0) \tag{25b}$$

of $W(k)$-modules. Let $\mathrm{Lie}(\mathcal{H}) = \oplus_{i=-1}^{i} \tilde{F}_0^i(\mathrm{Lie}(\mathcal{H}))$ be the direct sum decomposition such that $\mathbb{G}_m$ acts trough μ_0 on $\tilde{F}_0^i(\mathrm{Lie}(\mathcal{H}))$ via the $-i$-th power of the identity character of $\mathbb{G}_m$. Thus we have an identity

$$\tilde{F}_0^{-1}(\mathrm{Lie}(\mathcal{H})) = \mathrm{Hom}(F_0^1, F_0^0) \cap \mathrm{Lie}(\mathcal{H}), \tag{25c}$$

the intersection being taken inside $\mathrm{End}(M_0)$. Let $\mathcal{U}$ be the connected, smooth, unipotent subgroup scheme of $\mathcal{H}$ defined by the following rule: if C is a commutative $W(k)$-algebra, then $\mathcal{U}(C) = 1_{M_0 \otimes_{W(k)} C} + \tilde{F}_0^{-1}(\mathrm{Lie}(\mathcal{H})) \otimes_{W(k)} C$.

The smooth $k(v)$-scheme $\mathcal{L}(N)^{\mathrm{s}}_v$ is equidimensional of dimension d. As μ_0 belongs to ν_0 and thus to ν, from Formula (17) we get that the rank e_- of $\tilde{F}_0^{-1}(\mathrm{Lie}(\mathcal{H}))$ is precisely d. Thus the smooth $k(v)$-scheme $\mathcal{L}(N)^{\mathrm{s}}_v$ is equidimensional of dimension e_-. In other words, the axiom (i) of [Va5, Def. 4.2.1] holds.

Let R_y be the completion of the local ring of $\mathcal{N}(N)^{\mathrm{s}}_{W(k)}$ at its k-valued point defined by y. We fix an identification $R_y = W(k)[[x_1, \ldots, x_d]]$. Let Φ be the Frobenius lift of R_y which is compatible with σ_k and which takes x_i to x_i^p for all $i \in \{1, \ldots, d\}$. We have a natural morphism $\mathrm{Spec}(R_y) \to \mathcal{N}(N)^{\mathrm{s}}$ which is formally étale. The principally quasi-polarized filtered F-crystal over R_y/pR_y of the pull back to $\mathrm{Spec}(R_y)$ of $(\mathcal{V}, \Lambda_{\mathcal{V}})$ is isomorphic to

$$(M_0 \otimes_{W(k)} R_y, F_0^1 \otimes_{W(k)} R_y, h_y(g_y\phi_0 \otimes \Phi), \psi^*, \nabla_y), \tag{26a}$$

where $h_y \in \mathcal{H}(R_y)$ is such that modulo the ideal $(x_1, \ldots, x_d)$ of R_y is the identity element of $\mathcal{H}(W(k))$ and where ∇_y is an integrable, nilpotent modulo p connection on $M_0 \otimes_{W(k)} R_y$. We have:

(i) for each element $\alpha \in \mathcal{J}$, the tensor $t_\alpha \in \mathcal{T}(M_0[\frac{1}{p}]) \otimes_{B(k)} R_y[\frac{1}{p}] = \mathcal{T}(M_0 \otimes_{W(k)} R_y[\frac{1}{p}])$ is the de Rham realization of the pull back to $\mathrm{Spec}(R_y[\frac{1}{p}])$ of the Hodge cycle $v_\alpha^{\mathcal{V}}$ on $\mathcal{V}_{\mathbb{Q}}$ and therefore it is annihilated by ∇_y;

(ii) the connection ∇_y is versal.

The two properties (i) and (ii) hold as, up to $W(k)$-automorphisms of R_y that leave invariant its ideal $(x_1, \ldots, x_d)$, we can choose the morphism $h_y : \mathrm{Spec}(R_y) \to \mathcal{H}$ to factor through a formally étale morphism $h_y \to \mathcal{U}$ (i.e., we can choose h_y to be the universal element of the completion of $\mathcal{U}$). If $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme, then the fact that such a choice of h_y is possible follows from [Va1, Subsect. 5.4]. The general case is entirely the same (for instance, cf. [Va7, Subsects. 3.3 and 3.4]).

We recall the standard argument that ∇_y annihilates each t_α with $\alpha \in \mathcal{J}$. We view $\mathcal{T}(M_0)$ as a module over the Lie algebra (associated to) $\mathrm{End}(M_0)$ and accordingly we denote also by ∇_y the connection on $\mathcal{T}(M_0 \otimes_{W(k)} R_y[\frac{1}{p}])$ which extends naturally the connection ∇_y on $M_0 \otimes_{W(k)} R_y$. The Φ-linear action of $h_y(g_y\phi_0 \otimes \Phi)$ on $M_0 \otimes_{W(k)} R_y$ extends to a Φ-linear action of $h_y(g_y\phi_0 \otimes \Phi)$ on $\mathcal{T}(M_0 \otimes_{W(k)} R_y[\frac{1}{p}])$. For instance, if $a \in M_0^* \otimes_{W(k)} R_y = (M_0 \otimes_{W(k)} R_y)^*$ and if $b \in M_0 \otimes_{W(k)} R_y$, then $[h_y(g_y\phi_0 \otimes \Phi)](a) \in M_0^* \otimes_{W(k)} R_y[\frac{1}{p}]$ maps $[h_y(g_y\phi_0 \otimes \Phi)](b)$ to $\Phi(a(b))$. As ϕ_0, g_y, and h_y fix t_α, the tensor $t_\alpha \in \mathcal{T}(M_0 \otimes_{W(k)} R_y[\frac{1}{p}])$ is also fixed by $h_y(g_y\phi_0 \otimes \Phi)$. The connection ∇_y is the unique connection on $M_0 \otimes_{W(k)} R_y$ such that we have an identity

$$\nabla_y \circ [h_y(g_y\phi_0 \otimes \Phi)] = [h_y(g_y\phi_0 \otimes \Phi)] \otimes d\Phi) \circ \nabla_y,$$

cf. [Fa2, Thm. 10]. From the last two sentences we get that

$$\nabla_y(t_\alpha) = [h_y(g_y\phi_0 \otimes \Phi) \otimes d\Phi](\nabla_y(t_\alpha)).$$

As we have $d\Phi(x_i) = px_i^{p-1}dx_i$ for all $i \in \{1, \ldots, d\}$, by induction on $q \in \mathbb{N}$ we get that $\nabla_y(t_\alpha) \in \mathcal{T}(M_0) \otimes_{W(k)} (x_1, \ldots, x_d)^q \Omega^\wedge_{R_y/W(k)}[\frac{1}{p}]$. Here $\Omega^\wedge_{R_y/W(k)}$ is the p-adic completion of the sheaf of relative 1-differential forms. As R_y is complete with respect to the $(x_1, \ldots, x_d)$-topology, we have $\nabla_y(t_\alpha) = 0$.

Due to the property (ii), the morphism $\mathcal{L}(N)_v^{\mathrm{s}} \to \mathcal{A}_{d,1,N,k}$ induces k-epimorphisms at the level of complete, local rings of residue field k i.e., it is a formal closed embedding at all k-valued points (this is precisely the statement of [Va7, Part I, Thm. 1.5 (b)]). Thus the axiom (ii) of [Va5, Def. 4.2.1] holds.

Based on the property (i) and a standard application of Artin's approximation theorem, we get that there exists an étale map $\eta_y : \mathrm{Spec}(E_y) \to \mathcal{N}(N)^{\mathrm{s}}_{W(k)}$ whose image contains the k-valued point of $\mathcal{N}(N)^{\mathrm{s}}_{W(k)}$ defined naturally by y and for which the following three properties hold:

(iii) the p-adic completion $E_y^\wedge$ of E_y has a Frobenius lift Φ_{E_y};

(iv) the principally quasi-polarized filtered F-crystal over E_y/pE_y of the pull back to $\mathrm{Spec}(E_y/pE_y)$ of $(\mathcal{V}, \Lambda_{\mathcal{V}})$ is isomorphic to

$$(26b) \qquad (M_0 \otimes_{W(k)} E_y^\wedge, F_0^1 \otimes_{W(k)} E_y^\wedge, j_y(g_y\phi_0 \otimes \Phi_{E_y}), \psi^*, \nabla_y^{\mathrm{alg}}),$$

where $j_y \in \mathcal{H}(E_y^\wedge)$ and where ∇_y^{alg} is an integrable, nilpotent modulo p connection on $M_0 \otimes_{W(k)} R_y$ which is versal at each k-valued point of $\mathrm{Spec}(E_y^\wedge)$;

(v) for each element $\alpha \in \mathcal{J}$, the tensor $t_\alpha \in \mathcal{T}(M_0[\frac{1}{p}] \otimes_{B(k)} E_y^\wedge[\frac{1}{p}]) = \mathcal{T}(M_0 \otimes_{W(k)} E_y^\wedge[\frac{1}{p}])$ is the de Rham realization of the pull back to $\mathrm{Spec}(E_y^\wedge[\frac{1}{p}])$ of the Hodge cycle $v_\alpha^{\mathcal{V}}$ on $\mathcal{V}_{\mathbb{Q}}$ and therefore it is annihilated by ∇_y^{alg}.

Let $\bar{\eta}_y : \mathrm{Spec}(E_y/pE_y) \to \mathcal{L}(N)_{v,k}^{\mathrm{s}}$ be the étale map defined naturally by η_y. Let I_k be a finite set of k-valued points of $\mathcal{L}(N)_v^{\mathrm{s}}$ such that we have an identity

$$\cup_{\tilde{y} \in I_k} \mathrm{Im}(\bar{\eta}_{\tilde{y}}) = \mathcal{L}(N)_v^{\mathrm{s}}.$$

This means that the axiom (iii.a) of [Va5, Def. 4.2.1] holds for the family of étale maps $(\bar{\eta}_{\tilde{y}})_{\tilde{y} \in I_k}$.

Let $\mathcal{W}_{+0}$ be the maximal parabolic subgroup scheme of $\mathbf{GL}_{M_0}$ that normalizes F_0^1. Let $\mathcal{W}_{+0}^{\mathcal{H}} := \mathcal{H} \cap \mathcal{W}_{+0}$; it is a smooth subgroup scheme of $\mathcal{H}$ (cf. [Va5, Lem. 4.1.2]). As ∇_y^{alg} is versal at each k-valued point of $\mathrm{Spec}(E_y^\wedge)$, we have:

(vi) the reduction modulo p of j_y is a morphism $\mathrm{Spec}(E_y/pE_y) \to \mathcal{H}_k$ whose composite with the quotient morphism $\mathcal{H}_k \twoheadrightarrow \mathcal{H}_k/\mathcal{W}_{+0,k}^{\mathcal{H}}$ is étale.

Property (vi) implies that the axiom (iii.b) of [Va5, Def. 4.2.1] holds for $(\bar{\eta}_{\tilde{y}})_{\tilde{y} \in I_k}$.

Based on properties (iv) and (v), it is easy to see that the axiom (iii.c) of [Va5, Def. 4.2.1] holds for $(\bar{\eta}_{\tilde{y}})_{\tilde{y} \in I_k}$.

The fact that the axiom (iii.d) of [Va5, Def. 4.2.1] holds as well for $(\bar{\eta}_{\tilde{y}})_{\tilde{y} \in I_k}$ is only a particular case of Faltings' deformation theory [Fa, §7, Thm. 10 and Rm. i) to iii) after it], cf. the versality part of the property (iv). More precisely, if $\omega \in \mathrm{Ker}(\mathcal{H}(R_y) \twoheadrightarrow \mathcal{H}(R_y/(x_1, \ldots, x_d)))$ is such that the composite of ω modulo p with the quotient morphism $\mathcal{H}_k \twoheadrightarrow \mathcal{H}_k/\mathcal{W}_{+0,k}^{\mathcal{H}}$ is formally étale, then there exists an $W(k)$-automorphism $a_y : R_y \tilde{\to} R_y$ that leaves invariant the ideal $(x_1, \ldots, x_d)$ and for which the extension of (26a) via a_y is isomorphic to

$$(M_0 \otimes_{W(k)} R_y, F_0^1 \otimes_{W(k)} R_y, \omega(g_y\phi_0 \otimes \Phi), \psi^*, \nabla_y)$$

under an isomorphism defined by an element of $\mathrm{Ker}(\mathcal{H}(R_y) \twoheadrightarrow \mathcal{H}(R_y/(x_1, \ldots, x_d)))$. Thus axioms (i) to (iii) of [Va5, Def. 4.2.1] hold i.e., $\mathcal{L}(N)_v^{\mathrm{s}}$ is a quasi Shimura p-variety of Hodge type relative to $\mathcal{F}_0$ in the sense of [Va5, Def. 4.2.1]. $\square$

8.6. Level m stratification

We assume that properties (*) to (****) of this Section hold. Let m be a positive integer. From Theorem 9 and [Va5, Cor. 4.3] we get that there exists a stratification $\mathfrak{L}_m$ of $\mathcal{L}(N)_v^{\mathrm{s}}$ with the property that two geometric points $y_1, y_2 : \mathrm{Spec}(k) \to \mathcal{L}(N)_v^{\mathrm{s}}$ factor through the same stratum if and only $\mathcal{E}_{g_{y_1}}[p^m]$ is inner isomorphic to $\mathcal{E}_{g_{y_2}}[p^m]$. We call $\mathfrak{L}_m$ as the level m stratification of $\mathcal{L}(N)_v^{\mathrm{s}}$. Among its many properties we list here only three:

Proposition 6. *Let l be either $k(v)$ or an algebraically closed field of countable transcendental degree over $k(v)$. Let $\mathfrak{n}$ be a stratum of $\mathfrak{N}^{\mathrm{s}}$ which is a locally closed subscheme of $\mathcal{L}(N)^{\mathrm{s}}_{v,l}$. Then we have:*

(a) *there exists a family $(\mathfrak{l}_i)_{i\in L(\mathfrak{n})}$ of strata of $\mathfrak{L}_{\lceil\frac{r}{2}\rceil}$ which are locally closed subschemes of $\mathcal{L}(N)^{\mathrm{s}}_{v,l}$ and such that we have an identity*

$$\mathfrak{n}(\bar{l}) = \cup_{i\in L(\mathfrak{n})}\mathfrak{l}_i(\bar{l}); \tag{45}$$

(b) *the scheme $\mathfrak{n}$ is regular and equidimensional;*

(c) *the $\mathcal{L}(N)^{\mathrm{s}}_{v,l}$-scheme $\mathfrak{n}$ is quasi-affine.*

Proof: The Newton polygon of a p-divisible group D over k of codimension c and dimension d is uniquely determined by $D[p^{\lceil\frac{cd}{c+d}\rceil}]$, cf. [NV2, Thm. 1.2]. Thus the Newton polygon of (M,ϕ) is uniquely determined by the inner isomorphism class of $\mathcal{E}_{g_y}[p^{\lceil\frac{r}{2}\rceil}]$. From this the part (a) follows.

Parts (b) and (c) are particular cases of [Va5, Cor. 4.3]. $\square$

Remark 6. **(a)** *For PEL type Shimura varieties, the idea of level m stratifications shows up first in [We]. The level 1 stratifications generalize the Ekedahl–Oort stratifications studied extensively by Kraft, Ekedahl, Oort, Wedhorn, Moonen, and van der Geer.*

(b) *Suppose that $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme and that the properties (*) and (***) of this Section hold. As $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme, it is easy to see that the properties (**) and (****) hold as well. Thus the level m stratification $\mathfrak{L}_m$ exists. It is known that $\mathfrak{L}_1$ has a finite number of strata (see [Va10, Sect. 12]).*

8.6.1. Problem

Study when $\mathfrak{L}_m$ has the purity property.

8.7. Traverso stratifications

We continue to assume that properties (*) to (****) of this Section hold. Let

$$n_v \in \mathbb{N}$$

be the smallest positive integer such that for all elements $g \in \mathcal{H}(W(k))$ and $g_1 \in \mathrm{Ker}(\mathcal{H}(W(k)) \to \mathcal{H}(W_{n_v}(k)))$, the quadruples $\mathcal{E}_g$ and $\mathcal{E}_{gg_1}$ are isomorphic. The existence of n_v is implied by [Va3, Main Thm. A].

Lemma 1. *We assume that the assumptions (*) to (****) of this Section hold. Let $g_1, g_2 \in \mathcal{H}(W(k))$. The D-truncations of level m of $\mathcal{E}_{g_1}$ and $\mathcal{E}_{g_2}$ are inner isomorphic if and only if there exists $g_3 \in \mathcal{H}(W(k))$ such that we have $g_3g_2\phi_0g_3^{-1} = g_0g_1\phi_0$ for some element $g_0 \in \mathrm{Ker}(\mathcal{H}(W(k)) \to \mathcal{H}(W_m(k)))$.*

Proof: This is only a principal quasi-polarized variant of [Va3, Lem. 3.2.2]. Its proof is entirely the same as of loc. cit. $\square$

Due to Lemma 1, from the very definition of n_v we get that for every two elements $g_1, g_2 \in \mathcal{H}(W(k))$ we have the following equivalence:

(i) $\mathcal{E}_{g_1}$ is isomorphic to $\mathcal{E}_{g_2}$ if and only if $\mathcal{E}_{g_1}[p^{n_y}]$ is isomorphic to $\mathcal{E}_{g_2}[p^{n_y}]$.

Due to the property (i), for $m \geq n_v$ we have an identity

$$\mathfrak{L}_m = \mathfrak{L}_{n_v}.$$

We refer to

$$\mathfrak{T} := \mathfrak{L}_{n_v}$$

as the Traverso stratification of $\mathcal{L}(N)_v^{\mathrm{s}}$. Such stratifications were studied in [Tr1] to [Tr2] (using the language of group actions), in [Oo] (using the language of foliations), and in [Va3] and [Va5] (using the language of ultimate or Traverso stratifications). Based on Theorem 9, the next Theorem is only a particular case of [Va5, Cor. 4.3.1 (b)].

Theorem 10. *Under the assumptions (*) to (****) of this Section, the Traverso stratification $\mathfrak{T}$ of $\mathcal{L}(N)_v^{\mathrm{s}}$ has the purity property.*

8.7.1. Problems

1. Find upper bounds for n_v which are sharp.

2. Study the dependence of n_v on v.

8.7.2. Example

We assume that f is an isomorphism i.e., we have an identification $(G, \mathcal{X}) = (\mathbf{GSp}(W, \psi), \mathcal{S})$. We have $v = p$ and thus we will denote n_v by n_p. We also assume that y is a supersingular point i.e., all Newton polygon slopes of (M, ϕ) are $\frac{1}{2}$. The isomorphism class of (M, ϕ, ψ_M) is uniquely determined by $\mathcal{E}_{g_y}[p^r]$, cf. [NV1, Thm. 1.3]. Moreover, in general we can not replace in the previous sentence $\mathcal{E}_{g_y}[p^r]$ by $\mathcal{E}_{g_y}[p^{r-1}]$ (cf. [NV1, Example 3.3] and the result [Va3, Prop. 5.3.3] which says that each principally quasi-polarized Dieudonné module over k is the one attached to a principally polarized abelian variety over k).

Therefore, the restrictions of $\mathfrak{T}$ and $\mathfrak{L}_r$ to the (reduced) supersingular locus of $\mathcal{A}_{r,1,N,\mathbb{F}_p} = \mathcal{L}(N)_p = \mathcal{L}(N)_p^{\mathrm{s}}$ coincide and we have an inequality

$$n_p \geq r.$$

Based on Traverso's isomorphism conjecture (cf. [Tr3, §40, Conj. 4] or [NV1, Conj. 1.1]), one would be inclined to expect that n_p is in fact exactly r. However, we are not at all at the point where we could state this as a solid expectation.

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